# Collusion-proof mechanisms for multi-unit procurement

Martin Hagen*
CUNEF

January 23, 2023

**Abstract**

A principal wants to procure multiple homogeneous units from finitely many agents. Each agent has an increasing and convex cost function, whose exact shape is unknown to the principal. Utility is quasilinear in money. We study which mechanisms are strategy-proof and robust to collusion, both when the agents can exchange money and physical units (reallocation-proofness) and when they cannot (group strategy-proofness). To achieve reallocation-proofness, the principal must offer the agents a fixed price per unit. While group-strategy-proof mechanisms can be more complex, they are inefficient and run the risk of procuring no units at all. We characterize the set of group-strategy-proof and anonymous mechanisms with a uniform price. A standout feature is that the number of potential prices is bounded above by the number of agents.

**Keywords:** mechanism design, procurement, collusion, group strategy-proofness, reallocation-proofness.

**JEL Classification:** C72, D82.

---

*CUNEF Universidad, Calle Pirineos, 55, 28040 Madrid, Spain. Email: martin.hagen@cunef.edu. I am indebted to Ángel Hernando-Veciana for his guidance and advice. Many thanks to Salvador Barberà, Luis Corchón, Martin Dumav, Vitali Gretschko, Ruben Juarez, Flip Klijn, Antoine Loeper, Jordi Massó, Hervé Moulin, David Pérez-Castrillo, Carmelo Rodríguez-Álvarez, Antonio Romero-Medina, Johannes Schneider, three anonymous referees and the advisory editor for helpful comments. Financial support from the Spanish Ministry of Economy and Competitiveness, grant ECO2015-68406-P, during the early stages of this research is gratefully acknowledged.

## 1 Introduction

Procurement is a major part of economic activity, with public procurement alone accounting for 12 percent of global GDP (Bosio & Djankov, 2020). In electricity markets, business-to-business supply chains and many other cases, the buyer wishes to acquire multiple homogeneous units. To achieve low prices, procurement is often conducted by auction. There is plenty of empirical evidence, however, that auctions are prone to collusion: bidders compete less fiercely so as to drive up the price they receive (Marshall & Marx, 2012). This paper studies how multi-unit procurement mechanisms should be designed to prevent collusion.

In our model, a principal (female) procures discrete units of a homogeneous good from finitely many agents (male). Each agent's utility equals the payment he receives from the principal minus his cost of production, which is increasing and convex in the quantity provided. The principal designs a deterministic mechanism that specifies the quantities and payments for each possible realization of the agents' costs functions, whose exact shapes are unknown to the principal. There are no constraints on the amount of money the principal can spend or the total quantity to be procured.

The principal wants a mechanism that is strategy-proof and robust to collusion in the following sense: no group of agents who collectively deviate from their individually dominant strategies can make some of them better off without making any other group member worse off. Collusion-proofness comes in two forms, depending on whether the agents only deviate within the mechanism or also reallocate afterwards. The former refers to a "weak cartel", the latter to a "strong cartel" (McAfee & McMillan, 1992).

A mechanism that discourages strong cartels is said to be "reallocation-proof", that is, no group of agents ever benefit from a joint deviation, even if they exchange money or physical units. Theorem 1 shows that a mechanism is reallocation-proof if and only if it is a "fixed-price mechanism": the principal sets an exogenous unit price and each agent provides his optimal quantity at that price. A fixed-price mechanism has two unique features that make it reallocation-proof. First, the agents cannot affect each other's prices, which prevents them from bribing each other to misreport. Second, all marginal prices coincide, so there are no gains from arbitrage.

Side deals that involve money or physical units are relatively easy to detect for antitrust authorities. The agents may thus naturally refrain from reallocation, in which case cartels are weak. A collusion-resilient mechanism then merely needs to be "(strongly) group-strategy-proof", that is, if a coordinated misreport by a group of agents makes at least one of them better off, at least one other group member must become worse off.

We offer two impossibility results for group-strategy-proof mechanisms. First, if the agents face participation constraints, they may provide no quantity at all (Proposition 1). The explanation is that, by group strategy-proofness, the number of potential transfers for each agent is bounded above by the number of quantity profiles in the mechanism's range (Lemma 1). Due to this transfer rigidity, the principal is unable to compensate the agents financially when their costs become very large. For a similar reason, group-strategy-proof mechanisms are never efficient, that is, they cannot assign a given quantity to the agents with the lowest marginal costs (Proposition 2). This second result holds even without individual rationality.

On the positive side, group strategy-proofness does not require the price to be fixed (unlike reallocation-proofness). But how exactly do alternative mechanisms look like? We give a partial answer by characterizing the class of group-strategy-proof and anonymous mechanisms that pay all agents the same price per unit. This feature is inspired by the uniform-price auction, which is widely perceived as a fair way to assign multiple units because it eliminates "price envy" (Shinozaki, 2022) among the agents. In contrast to a fixed-price mechanism, the level of the uniform price may be endogenous.

Theorem 2 shows that every group-strategy-proof, anonymous uniform-price mechanism can be implemented as follows: With $n \in \mathbb{N}$ agents, the principal exogenously specifies $n$ prices (which can, but need not, be different). She starts by asking each agent whether he would be willing to provide at least one unit at the largest price, $p(n)$, which is also the $n$-th lowest price. All agents who refuse exit the mechanism, providing no quantity and receiving no payment. The other $m$ agents are then offered the $m$-th lowest price, $p(m)$. Since $p(m) \leq p(n)$, some of the agents who "participated" before may stop to do so now. They exit as well, and the price is lowered accordingly. Once every still-active agent is willing to provide at least one unit, we have reached the final price. At this price, say $p(l)$, the $l$ remaining agents supply their optimal quantities.

The reason why these "$n$-price mechanisms" are group strategy-proof is simple: A group of agents can gain from a joint deviation only if the final price rises. To this end, at least one agent who truthfully would exit the mechanism must stay active. This agent becomes worse off because he provides units at a price below his cost. The converse statement—there do not exist other group-strategy-proof, anonymous mechanisms with a uniform price—is not obvious. Its proof is a major part of our analysis.

If all $n$ prices coincide, we get a fixed-price mechanism. Is it ever beneficial for the principal to offer more than one price? Suppose she derives a constant value from each unit procured and maximizes her expected total value net of the transfers paid to the agents. Proposition 3 shows that, if the agents' costs are independent and identically distributed (iid), the principal's optimal $n$-price mechanism has a single price. Simply put, an $n$-price mechanism allows the principal to offer different prices to different expected subsets of agents, namely those who remain active in each step. With iid costs, all these subsets consist of "average" agents. Thus, optimally, the principal chooses the same price for each of them.

Our results have natural analogues when homogeneous units are sold to agents with increasing and concave value functions. Only Proposition 1 does not carry over because a constant mechanism that charges zero for each unit trivially satisfies group strategy-proofness and individual rationality. A special case of this reverse model is that the agents share the cost of producing the physical units. While our results still apply, they are arguably less pertinent to cost-sharing. One reason is that in many practical applications, like connecting municipalities to a water network (Young, 1994), it is physically impossible for the agents to trade quantity, undermining the case for reallocation-proofness. Moreover, the production cost is often determined by technical constraints; the natural requirement of budget balance then likely clashes with uniform pricing.

## 2 Related literature

The early work on strategy-proof mechanism design showed that allocative efficiency is unattainable if the agents can collude with side payments (Bennett & Conn, 1977; Green & Laffont, 1979).[1] Schummer (2000b) intro-

[1] One way out of this impossibility result is to assume that each agent belongs to a fixed coalition, which is unknown to the designer. In this case, the VCG mechanism can

duced the related concept of "bribe-proofness": if one agent pays another to misreport, they must not both become better off. Bribe-proofness usually leads to constant, posted-price or otherwise rigid mechanisms (Bu, 2016; Goldberg & Hartline, 2005; Mihalák et al., 2016; Mizukami, 2003; Schummer, 2000a, 2000b). Our model is no exception: under bribe-proofness, the agents' equilibrium payoffs are mutually independent (Lemma A.1), implying that each agent faces an exogenous price schedule and picks his optimal quantity (Lemma A.2).[2] Reallocation-proofness strengthens bribe-proofness by discouraging collusion that involves the exchange of not just money but also physical units. According to our Theorem 1, reallocation-proofness characterizes the subclass of bribe-proof mechanisms that do not price-discriminate in any way.[3]

Collusion is easier to deter if the agents cannot reallocate. We study the strong version of group strategy-proofness, which rules out joint misreports that make some agents better off and none worse off. Our results would not hold under mere strategy-proofness (cf. Barberà et al., 2010, 2016; Le Breton & Zaporozhets, 2009). Moreover, in our model, group strategy-proofness is logically independent of non-bossiness. Introduced by Satterthwaite and Sonnenschein (1981), non-bossiness says that if an agent changes his report but still receives the same allocation, then all other agents' allocations should also remain the same. Due to the availability of money, an agent can be indifferent between multiple allocations; which one is selected may depend on the other agents' reports. For this reason, the fixed-price mechanism that we define in Section 5.1 is potentially bossy yet group strategy-proof. Conversely, there exist mechanisms that are strategy-proof and non-bossy but not group strategy-proof (Hashimoto & Saitoh, 2015; Mutuswami, 2005). For an extensive discussion of non-bossiness and its relationship to group strategy-proofness, see Thomson (2016).

Group strategy-proofness clashes with allocative efficiency (Proposition 2).

---

be made collusion-proof by asking the agents to report not only their preferences but also their coalitions (Chen & Micali, 2012; Deckelbaum & Micali, 2017; Gorokh et al., 2019).

[2] Mechanisms with exogenous prices can also arise under conditions weaker than bribe-proofness, such as strategy-proofness plus individual rationality and budget balance. Characterizations of this type are available for housing markets (Miyagawa, 2001), public-good provision (Bierbrauer & Hellwig, 2016; Drexl & Kleiner, 2018; Kuzmics & Steg, 2017), bilateral trade (Barberà & Jackson, 1995; Hagen & Hernando-Veciana, 2021; Hagerty & Rogerson, 1987) and land assembly (Grossman et al., 2019), among others.

[3] Notions similar to reallocation-proofness, but in models without monetary transfers, are studied by Massó and Neme (2007) and Pápai (2000).

This impossibility result motivates a weaker version which considers only those joint misreports to be problematic that make *all* conspiring agents better off. Weak group strategy-proofness, coupled with efficiency or other properties, has been studied in several models with money, such as queuing (Chun et al., 2014; Mitra & Mutuswami, 2011; Mukherjee, 2013) or the allocation of a single unit (Mukherjee, 2014, 2020). Given the combination of strategy-proofness and efficiency, these papers usually axiomatize a sub-class of VCG mechanisms (Holmström, 1979).

VCG mechanisms are also the focus of the literature on "fair imposition" (Porter et al., 2004), where a *fixed* number of costly tasks, plus money, must be assigned to the agents. In our model, this objective is incompatible with group strategy-proofness and efficiency (Proposition 2). Other papers invoke strategy-proofness and efficiency together with $k$-fairness (Atlamaz & Yengin, 2008; Porter et al., 2004), egalitarian equivalence (Yengin, 2012, 2017) or the identical-preferences lower bound (Yengin, 2013). To which extent these fairness properties can be satisfied by group-strategy-proof, and thus non-efficient, mechanisms is an open question.

The $n$-price mechanisms identified by our Theorem 2 have the feature that the price weakly increases in the number of agents willing to supply a positive quantity. Mechanisms of this type are known as "cross-monotonic" in the literature on group-strategy-proof cost sharing, which was initiated by Moulin (1999). Assuming each agent demands at most one unit of output and the production cost is submodular, he showed that a mechanism satisfies group strategy-proofness and three auxiliary properties if and only if it is cross-monotonic. Similar equivalences were established by Immorlica et al. (2008) and Juarez (2013) when the cost function is unrestricted but a tie-breaking rule governs cases of indifference. Doing away with this tie-breaking rule, Pountourakis and Vidali (2012) completely characterize group-strategy-proof mechanisms via a generalization of cross-monotonicity. Roughgarden and Sundararajan (2009) analyze the trade-off between budget balance and efficiency for cross-monotonic mechanisms. Moulin and Shenker (2001) and Juarez (2018) identify optimal mechanisms among those satisfying cross-monotonicity.

Few papers allow the agents to demand or supply multiple units. Moulin (1999) shows that, when the production cost is supermodular, group strategy-proofness requires the agents to be served one after another; but these "incremental" mechanisms violate standard fairness notions. Mehta et al. (2009) propose a generalization of cross-monotonic mechanisms that work for both single- and multi-unit demand; they are weakly but not strongly group strategy-proof. By contrast, we characterize strongly group-strategy-proof and "fair" mechanisms in a model with multi-unit supply. Our uniform-price condition is logically unrelated to sub- and supermodularity.[4]

# 3 Model

A principal (female) seeks to procure discrete homogeneous units from $n \in \{2, 3, \dots\}$ agents (male). An allocation $a_i \coloneqq (q_i, t_i)$ for agent $i \in N \coloneqq \{1, 2, \dots, n\}$ specifies the quantity $q_i \in \mathbb{N} \coloneqq \{0, 1, \dots\}$ that $i$ provides and the monetary transfer $t_i \in \mathbb{R}$ that $i$ receives. For any group $G \subseteq N$, define $a_G \coloneqq (a_i)_{i \in G}$, $q_G \coloneqq (q_i)_{i \in G}$ and $t_G \coloneqq (t_i)_{i \in G}$.

Agent $i$'s total cost of providing $q_i$ units is denoted by $c_i(q_i) \geq 0$, his marginal cost by $\Delta c_i(q_i) \coloneqq c_i(q_i) - c_i(q_i - 1)$. The domain $C$ contains all cost functions $c\colon \mathbb{N} \to \mathbb{R}_+ \coloneqq [0, \infty)$ such that $c(0) = 0$, $0 < \Delta c(1) < \Delta c(2) < \cdots$ and $\lim_{q \to \infty} \Delta c(q) = \infty$. In words, cost functions are normalized to zero at the origin, (strictly) increasing, (strictly) convex and eventually outgrow any linear function.[5] While it is common knowledge that $c_i \in C$ for all $i \in N$, the principal does not know the exact shape of $c_i$.

For all cost profiles $(c_i)_{i \in N} \in C^n$ and groups $G \subseteq N$ of size $g = |G|$, define $c_G \coloneqq (c_i)_{i \in G} \in C^g$ and $c_{-G} \coloneqq (c_i)_{i \in N \setminus G} \in C^{n-g}$. Whenever $c_i = c_j$ for all $i, j \in G$, we write $c_G \in C^g_*$. Moreover, for all $c_G, \hat{c}_G \in C^g$, $c_G > \hat{c}_G$ means that $c_i(q_i) > \hat{c}_i(q_i)$ for all $i \in G$ and $q_i \in \mathbb{N} \setminus \{0\}$.

The utility that agent $i \in N$ with cost function $c_i \in C$ derives from allocation $(q_i, t_i) \in \mathbb{N} \times \mathbb{R}$ is $t_i - c_i(q_i)$. His optima in a set of allocations $A_i \subseteq \mathbb{N} \times \mathbb{R}$ are denoted by $\mathrm{Opt}(c_i, A_i) \coloneqq \arg\max_{(q_i, t_i) \in A_i} \{t_i - c_i(q_i)\}$.

[4] Barberà and Jackson (1995) identify mechanisms similar to ours in an exchange economy with classical preferences, invoking the properties of strategy-proofness, individual rationality, non-bossiness, anonymity and tie-freeness.

[5] The limit condition is only needed for the mechanisms in Section 5 to be well defined. Alternatively, we could put an upper bound on each $q_i$.

# 4 Definitions

In a (deterministic, direct) mechanism, the agents report cost functions $c_N \in C^n$ to the principal, who then assigns a quantity $\varphi_i(c_N) \in \mathbb{N}$ and a transfer $\tau_i(c_N) \in \mathbb{R}$ to each $i \in N$.

**Definition 1.** A **mechanism** $(\varphi_i, \tau_i)_{i\in N}$ is a mapping from $C^n$ to $(\mathbb{N}\times\mathbb{R})^n$.

Agent $i$'s allocation function is denoted by $\alpha_i \coloneqq (\varphi_i, \tau_i)$. For all groups $G \subseteq N$, define $\alpha_G \coloneqq (\alpha_i)_{i\in G}$, $\varphi_G \coloneqq (\varphi_i)_{i\in G}$ and $\tau_G \coloneqq (\tau_i)_{i\in G}$. If the mechanism under consideration is clear, $u_i(\hat{c}_N|c_i) \coloneqq \tau_i(\hat{c}_N) - c_i(\varphi_i(\hat{c}_N))$ represents $i$'s utility when his true cost function is $c_i$ and the reported cost profile is $\hat{c}_N$. If $i$ is truthful at $\hat{c}_N$ (i.e. $\hat{c}_i = c_i$), we write $u_i^*(\hat{c}_N) \coloneqq u_i(\hat{c}_N|c_i)$.

We now introduce three strategic properties. First, a mechanism is strategy-proof if no agent ever benefits from misreporting his cost function.

**Definition 2.** $(\varphi_i, \tau_i)_{i\in N}$ is **strategy-proof** if for all $c_N \in C^n$, $i \in N$ and $\hat{c}_i \in C$, $u_i^*(c_N) \geq u_i(\hat{c}_i, c_{-i}|c_i)$.

Group strategy-proofness extends the truth-telling incentives from individuals to groups. Specifically, if a joint misreport by a group of agents makes some of them better off, at least one other group member must become worse off. The latter should thus object to this collusive arrangement.

**Definition 3.** $(\varphi_i, \tau_i)_{i\in N}$ is **group strategy-proof** if for all $c_N \in C^n$, $G \subseteq N$ and $\hat{c}_G \in C^{|G|}$,

$$\exists i \in G : u_i^*(c_N) < u_i(\hat{c}_G, c_{-G}|c_i) \implies \exists j \in G : u_j^*(c_N) > u_j(\hat{c}_G, c_{-G}|c_j).$$

In a group-strategy-proof mechanism, joint deviations may still be tempting if the involved agents can exchange money or physical units among each other to compensate those group members who would otherwise suffer from the misreport. Reallocation-proofness holds if such side deals never pay off.

**Definition 4.** $(\varphi_i, \tau_i)_{i\in N}$ is **reallocation-proof** if for all $c_N \in C^n$, $G \subseteq N$, $\hat{c}_G \in C^{|G|}$ and $(\hat{q}_i, \hat{t}_i)_{i\in G} \in (\mathbb{N}\times\mathbb{R})^{|G|}$ with $\sum_{i\in G}\big[\hat{q}_i - \varphi_i(\hat{c}_G, c_{-G})\big] = 0 = \sum_{i\in G}\big[\hat{t}_i - \tau_i(\hat{c}_G, c_{-G})\big]$,

$$\exists i \in G : u_i^*(c_N) < \hat{t}_i - c_i(\hat{q}_i) \implies \exists j \in G : u_j^*(c_N) > \hat{t}_j - c_j(\hat{q}_j)$$

or, equivalently, $\sum_{i\in G} u_i^*(c_N) \geq \sum_{i\in G}\big[\hat{t}_i - c_i(\hat{q}_i)\big]$.

Note that reallocation-proofness implies group strategy-proofness, which in turn implies strategy-proofness.

Three further properties will be used in our analysis: Individual rationality says that no agent can be forced to participate in the mechanism. Efficiency holds if a given quantity is allocated to the agents with the lowest costs. Anonymity requires any permutation of the agents' cost functions to swap their utilities accordingly.

**Definition 5.** $(\varphi_i, \tau_i)_{i\in N}$ is **individually rational** if for all $c_N \in C^n$ and $i \in N$, $u_i^*(c_N) \geq 0$.

**Definition 6.** $(\varphi_i, \tau_i)_{i\in N}$ is $q^*$**-efficient** for $q^* \in \{1, 2, \dots\}$ if for all $c_N \in C^n$, $\varphi_N(c_N) \in \arg\min_{q_N \in \mathbb{N}^n \text{s.t.} \sum_{i\in N} q_i = q^*} \sum_{i\in N} c_i(q_i)$.

**Definition 7.** $(\varphi_i, \tau_i)_{i\in N}$ is **anonymous** if for all bijections $b\colon N \to N$ and all $c_N, \hat{c}_N \in C^n$ with $c_i = \hat{c}_{b(i)}$ for all $i \in N$, $u_i^*(c_N) = u_{b(i)}^*(\hat{c}_N)$ for all $i \in N$.

# 5 Results

We now present our results on reallocation-proofness (Section 5.1), followed by those on group strategy-proofness (Section 5.2).

## 5.1 Reallocation-proofness

If the agents can reallocate, only "fixed-price mechanisms" are robust to collusion. In a fixed-price mechanism, the principal offers a fixed price $p \geq 0$ per unit, and each agent supplies his optimal quantity at that price.[6] In addition, the principal can make or charge lump-sum transfers to the agents.

**Definition 8.** $(\varphi_i, \tau_i)_{i\in N}$ is a **fixed-price mechanism** if there exist a price $p \geq 0$ and transfers $(\underline{t}_i)_{i\in N} \in \mathbb{R}^n$ such that for all $c_N \in C^n$ and $i \in N$,

$$\varphi_i(c_N) \in \arg\max_{q_i \in \mathbb{N}} \big\{q_i p - c_i(q_i)\big\}, \tag{1}$$

$$\tau_i(c_N) = \varphi_i(c_N)p + \underline{t}_i. \tag{2}$$

**Theorem 1.** $(\varphi_i, \tau_i)_{i\in N}$ *is reallocation-proof if and only if it is a fixed-price mechanism.*

[6] An optimal quantity exists because $\lim_{q_i\to\infty} \Delta c_i(q_i) = \infty$ for all $i \in N$ and $c_i \in C$.

The proof is in Appendix A.1. While the "if" is rather straightforward, the "only if" has two main steps. First, we let the colluding agents exchange money ("bribes") but not physical units. Schummer's (2000b) insight that a "bribe-proof" mechanism must make each agent's equilibrium utility independent of the other agents' reports extends to our model. Every agent thus faces an exogenous set of allocations, from which he picks his optimum. In the second step, we allow the colluding agents to exchange physical units as well. Reallocation-proofness requires all units to be priced equally. Otherwise, two agents with different prices could benefit from arbitrage; that is, the agent with the higher price could first misreport to provide additional units to the principal and then pay the other agent to supply some of these units in his stead. The unique type of mechanism in which prices are both exogenous and equal is a fixed-price mechanism.

Another way of putting Theorem 1 is that, to prevent reallocation, the agents' marginal costs must be equalized. Hence, constant mechanisms are generally not reallocation-proof. Constancy means that $\alpha_N(c_N) = \alpha_N(\hat{c}_N)$ for all $c_N, \hat{c}_N \in C^n$. In any constant mechanism that procures a non-zero quantity, there exist a cost profile $c_N \in C^n$ and two agents $i, j \in N$ such that $\Delta c_i(\varphi_i(c_N)+1) < \Delta c_j(\varphi_j(c_N))$. Both agents gain if $j$ gives one unit to $i$, together with a payment $t$ satisfying $\Delta c_i(\varphi_i(c_N)+1) < t < \Delta c_j(\varphi_j(c_N))$, so reallocation-proofness is violated.[7] One could argue that this type of collusion need not be deterred because it improves efficiency without affecting the principal's expenditures. However, if she is unsure whether the agents will actually collude, then constant mechanisms are certainly unappealing. We will come back to this issue in the conclusion.

## 5.2 Group strategy-proofness

### 5.2.1 Two impossibility results

In a fixed-price mechanism, the total quantity procured is endogenous and can in fact be zero. This is an important drawback in practice because, say, a car manufacturer may need a certain number of wheel bearings from its suppliers. We now show that all group-strategy-proof and individually rational mechanisms suffer from this problem as well (Proposition 1). Moreover,

[7] There is one exception: a constant mechanism with $\varphi_i(c_N) = 0$ for all $c_N \in C^n$ and $i \in N$ amounts to a fixed-price mechanism with $p = 0$ and is thus reallocation-proof.

group strategy-proofness clashes with efficiency (Proposition 2).

Both impossibility results have the same root: group strategy-proofness restricts the flexibility of the principal's transfers. More precisely, strategy-proofness implies that if an agent provides the same quantity at two distinct reports, then he must also get the same transfer; otherwise, he will lie when he is paid less. Under group strategy-proofness, this result extends from individuals to groups. That is, any joint misreport that does not affect the quantities of the involved agents also leaves their transfers unchanged.

**Lemma 1.** *If $(\varphi_i, \tau_i)_{i\in N}$ is group strategy-proof, then for all $c_N \in C^n$, $G \subseteq N$ and $\hat{c}_G \in C^{|G|}$, $\left[\varphi_G(c_N) = \varphi_G(\hat{c}_G, c_{-G}) \implies \tau_G(c_N) = \tau_G(\hat{c}_G, c_{-G})\right]$.*

*Proof.* Lemma 1 is equivalent to Schummer's (2000a) Theorem 3. His model involves heterogeneous objects and agents with unit demand, but his arguments do not exploit these particular features and thus extend to our setting. For completeness, we offer our version of the proof in Supplement B.2.[8] □

We now explain why the principal may end up with no units at all. For ease of exposition, suppose each agent faces a hard capacity constraint, that is, an upper bound on the quantity he can possibly provide.[9] There are then finitely many quantity profiles $q_N \in \mathbb{N}^n$ in the mechanism's range. By Lemma 1, every $q_N$ is associated with a unique transfer profile $t_N \in \mathbb{R}^n$. It follows that each agent $i \in N$ may receive only finitely many transfers, with maximum $\bar{t}_i := \max\{t_i \in \mathbb{R} : \exists c_N \in C^n \text{ s.t. } \tau_i(c_N) = t_i\}$. Let $c_i \in C$ be such that $c_i(q_i) > \bar{t}_i$ for all $q_i \in \mathbb{N} \setminus \{0\}$, so any non-zero quantity yields negative utility. By individual rationality, $\varphi_i(c_i, \cdot) = 0$. Hence, each agent provides zero units when his costs are sufficiently large.

**Proposition 1.** *If $(\varphi_i, \tau_i)_{i\in N}$ is group strategy-proof and individually rational, then there exists $\hat{c}_N \in C^n$ such that for all $c_N \in C^n$ with $c_N > \hat{c}_N$, $\varphi_i(c_N) = 0$ for all $i \in N$.*

Without hard capacity constraints, the simple argument from above fails because the range of transfers can be unbounded, so $\bar{t}_i$ may not exist. Agent

[8]Results similar to Lemma 1 also appear in the literature on group-strategy-proof cost sharing (e.g. Immorlica et al., 2008; Juarez, 2013; Moulin, 1999).

[9]The model described in Section 3 accommodates "soft" capacity constraints in that diminishing capacity is captured by increasing costs.

$i$ may then not have a report that always triggers zero quantity.[10] The proof of Proposition 1 in Appendix A.2 deals with this technical difficulty.

For all $q^* \in \{1, 2, \dots\}$, Proposition 1 implies that no group-strategy-proof and individually rational mechanism is $q^*$-efficient because less than $q^*$ units are procured at some cost profiles. In fact, this conflict with efficiency persists when individual rationality is not imposed.

**Proposition 2.** *Consider any $q^* \in \{1, 2, \dots\}$. If $(\varphi_i, \tau_i)_{i \in N}$ is group strategy-proof, then it is not $q^*$-efficient.*

Simply put, the intuition is that efficiency requires the transfers to flexibly adjust to the agents' reported costs—which is at odds with the rigidity needed to deter group deviations (Lemma 1). The proof is in Appendix A.3.

### 5.2.2 Uniform-price mechanisms

Despite the impossibility results above, group strategy-proofness allows the principal more design flexibility than merely fixing a price. How exactly do alternative mechanisms look like? To address this question, we focus on mechanisms with a uniform price.

**Definition 9.** $(\varphi_i, \tau_i)_{i \in N}$ is a **uniform-price mechanism** if there exists a function $\pi \colon C^n \to \mathbb{R}$ such that for all $c_N \in C^n$ and $i \in N$,

$$\varphi_i(c_N) \in \arg\max_{q_i \in \mathbb{N}} \big\{ q_i \pi(c_N) - c_i(q_i) \big\}, \tag{3}$$

$$\tau_i(c_N) = \varphi_i(c_N)\pi(c_N). \tag{4}$$

(4) says that all agents receive the same price per unit. Moreover, by (3), their quantities are optimal at the realized price. These features are reminiscent of a uniform-price auction, which is widely perceived as a "fair" way to assign multiple units (e.g. Ausubel et al., 2014, p. 1393; Brenner et al., 2009, p. 272; Burkett and Woodward, 2020, p. 2). In practice, uniform-price auctions are used to allocate government bonds, electricity, pollution permits, etc. According to Milgrom (2004, p. 256), one reason for their popularity is the elimination of "price risk" across agents: since everyone's price is the same, corporate bidders avoid the uncomfortable situation of

[10] In other words, group strategy-proofness and individual rationality do *not* imply "consumer sovereignty" (Moulin, 1999, p. 290) with respect to quantity zero.

having to justify to their superiors or shareholders why they paid more or got paid less than other bidders. This argument can be formalized via the notion of "no price envy" (Shinozaki, 2022).

**Definition 10.** $(\varphi_i, \tau_i)_{i \in N}$ satisfies **no price envy** if for all $c_N \in C^n$, $i \in N$, $q_i \in \mathbb{N}$ and $j \in N$ with $\varphi_j(c_N) > 0$, $u_i^*(c_N) \geq q_i\big[\tau_j(c_N)/\varphi_j(c_N)\big] - c_i(q_i)$.

No price envy says that agent $i$ weakly prefers his allocation to supplying any quantity $q_i$ at agent $j$'s unit price $\tau_j(c_N)/\varphi_j(c_N)$. In Appendix A.4, we show that only uniform-price mechanisms satisfy no price envy and the auxiliary property of "zero transfer for zero quantity".

**Definition 11.** $(\varphi_i, \tau_i)_{i \in N}$ satisfies **zero transfer for zero quantity** if for all $c_N \in C^n$ and $i \in N$, $\big[\varphi_i(c_N) = 0 \implies \tau_i(c_N) = 0\big]$.

**Lemma 2.** *$(\varphi_i, \tau_i)_{i \in N}$ satisfies no price envy and zero transfer for zero quantity if and only if it is a uniform-price mechanism.*

Every uniform-price mechanism that is group strategy-proof and anonymous can be implemented as follows: The principal announces a sequence of prices $0 = p(0) \leq p(1) \leq \cdots \leq p(n) < \infty$. Each agent is asked whether he would "participate"—that is, provide at least one unit—at price $p(n)$. If all $n$ agents say yes, $p(n)$ is the final price. Otherwise, the agents who refuse to participate exit the mechanism; they supply no units and receive no payment. The $m \in \{0, 1, \ldots, n-1\}$ remaining agents are then asked whether they would still participate at price $p(m) \leq p(n)$. If all of them say yes, $p(m)$ is the final price. Otherwise, the naysayers are excluded, and the $l \in \{0, 1, \ldots, m-1\}$ remaining agents are offered price $p(l) \leq p(m)$. This process continues until a price is found at which all remaining agents participate.[11] Each of them supplies his optimal quantity at the final price. The direct version of this "$n$-price mechanism" is defined as follows.

**Definition 12.** $(\varphi_i, \tau_i)_{i \in N}$ is an ***n*-price mechanism** if there exist prices $0 = p(0) \leq p(1) \leq \cdots \leq p(n) < \infty$ such that for all $c_N \in C^n$ and $i \in N$,

$$\varphi_i(c_N) = \max\Big\{\arg\max_{q_i \in \mathbb{N}} \Big\{q_i p\big(\mu(c_N)\big) - c_i(q_i)\Big\}\Big\}, \tag{5}$$
$$\tau_i(c_N) = \varphi_i(c_N) p\big(\mu(c_N)\big),$$

[11] Such a price exists because zero agents participate at price $p(0) = 0$.

where $\mu(c_N) := \max\Big\{m \in \{0,1,\dots,n\} : \big|\{i \in N : c_i(1) \leq p(m)\}\big| = m\Big\}$.

**Definition 13.** $(\varphi_i, \tau_i)_{i\in N}$ and $(\hat{\varphi}_i, \hat{\tau}_i)_{i\in N}$ are **utility-equivalent** if for all $c_N \in C^n$ and $i \in N$, $\tau_i(c_N) - c_i(\varphi_i(c_N)) = \hat{\tau}_i(c_N) - c_i(\hat{\varphi}_i(c_N))$.

**Theorem 2.** *a) Every $n$-price mechanism is a group-strategy-proof, anonymous uniform-price mechanism. b) Every group-strategy-proof, anonymous uniform-price mechanism is utility-equivalent to an $n$-price mechanism.*

The proof is in Appendix A.5. To provide a better understanding of $n$-price mechanisms, let us briefly discuss some of their most notable features.

First, the price can only vary when some agents "drop out", that is, when they reject the current price and thus exit the mechanism. This feature is necessary for strategy-proofness. If an agent could influence the price he himself receives, then he would sometimes face incentives to misreport.

Second, the price can only decrease. This feature is again due to strategy-proofness. To see why, suppose an agent is unwilling to provide any units at the current price, but only slightly so. If this agent believes that some other agent will drop out and the principal will offer a higher price in response, then it will be better to stay active. Hence, remaining in the mechanism as long as the current price is at least as large as the agent's cost for the first unit is not a dominant strategy anymore.

Third, by how much the price decreases when an agent exits the mechanism is exogenously given. Hence, no agent can strategically drop out to keep the price high for the remaining agents. If he could, there would be opportunities for collusion, violating group strategy-proofness.

Finally, the reason why part b) of Theorem 2 holds only in utility-equivalent terms is the "max" operator in (5). Without it, $n$-price mechanisms can fail to be group strategy-proof. For example, suppose $n = 2$ and $c_1(1) < p(1) < p(2) = \hat{c}_2(1) < c_2(1)$. Then $\mu(c_1, c_2) = 1$ and $\mu(c_1, \hat{c}_2) = 2$. If $\varphi_2(c_1, \hat{c}_2) = 0$, group strategy-proofness would be violated at $(c_1, c_2)$ because $u_1^*(c_1, c_2) < u_1^*(c_1, \hat{c}_2)$ and $u_2^*(c_1, c_2) = u_2(c_1, \hat{c}_2|c_2)$. By choosing the largest optimum in cases of indifference, $n$-price mechanisms become group strategy-proof because for all $c_N \in C^n$ and $i \in N$, $\big[c_i(1) = p(\mu(c_N)) \implies \varphi_i(c_N) > 0\big]$. However, not all violations of this condition are problematic, such as $\varphi_2(\hat{c}_1, \hat{c}_2) = 0$ for $\hat{c}_1 = \hat{c}_2$ in the example above.

#### 5.2.3 On the optimal $n$-price mechanism

An $n$-price mechanism becomes a fixed-price mechanism if $p(1) = \ldots = p(n)$. Does the principal actually gain from having $n$ prices instead of one? To address this question, we must specify her objective function. Suppose the principal derives a constant value $v > 0$ from each unit procured and chooses a mechanism $(\varphi_i, \tau_i)_{i\in N}$ to maximize her expected value net of the transfers paid to the agents:

$$\mathbb{E}\left[v \sum_{i\in N} \varphi_i(c_N) - \sum_{i\in N} \tau_i(c_N)\right], \tag{6}$$

where the expectation is taken over $c_N \in C^n$. There is a natural trade-off: higher prices induce the agents to supply more quantity but decrease the principal's payoff per unit supplied. Under a standard assumption on the principal's beliefs, the best she can do is fix a single price.

**Proposition 3.** *Suppose the agents' costs are independent and identically distributed. Then, for every n-price mechanism, there exists a fixed-price mechanism that weakly increases the principal's payoff, as given by* (6)*.*

The proof is in Appendix A.6. To grasp the intuition, suppose the best fixed-price mechanism for the principal has price $p^*$.[12] We start by determining the optimal $p(1)$. This price is only relevant when $p(n)$ to $p(2)$ have fallen through, so a single agent is left; call him $i$. Since the agents' costs are independent, the fact that $n-1$ agents have dropped out does not give the principal any new information about $i$'s cost. And since the agents are ex-ante identical, they are all equally likely to be $i$. Hence, the optimal $p(1)$ is simply the optimal fixed price for the average agent, which coincides with $p^*$. Inductive application of this argument yields $p(1) = \ldots = p(n) = p^*$.

If the agents are asymmetric, however, the flexibility of $n$-price mechanisms can be beneficial to the principal. The reason is that an agent with relatively low expected costs is more likely to be the single agent to whom $p(1)$ is offered. The optimal fixed price for this "positively selected" agent, $p(1)$, may thus differ from the optimal fixed price for the entire set of agents, $p^*$. In other words, the principal can tailor the $n$ prices to the expected identities of the agents who survive each step of the $n$-price mechanism. For a numerical example that illustrates this idea, see Supplement B.4.

[12] The existence of such a $p^*$ is not assumed in the formal proof (Appendix A.6).

## 6 Conclusion

This paper has studied the design of multi-unit procurement mechanisms that are strategy-proof and robust to collusion. We have shown that strong cartels, whose members are able to reallocate among each other, can only be deterred with a fixed price per unit. Preventing weak cartels, whose members are unable to reallocate, is easier for the principal, but she must permit the procured quantity to be endogenous. If she wishes to pay a uniform price per unit, then the cardinality of the price range is bounded above by the number of agents. With independent and identically distributed cost functions, fixing a single price is optimal for the principal.

Our focus on uniform-price mechanisms appears desirable in practice because they are simple, transparent and eliminate price risk across agents. But mechanisms may still be perceived as "fair" if they charge different prices for different units or, even, to different agents. A characterization of all group-strategy-proof, individually rational, anonymous and/or envy-free mechanisms seems a worthwhile, yet daunting, endeavor for future research.

Both reallocation- and group strategy-proofness take the axiomatic stance that collusion is "bad" and must be prevented. On the downside, the resulting mechanisms are rather inflexible. The designer may do better by tolerating some forms of collusion. Optimal mechanisms that are robust to, but do not categorically rule out, collusion have been studied in the literature on Bayesian mechanism design (e.g. Che & Kim, 2006, 2009; Laffont & Martimort, 1997, 2000). An intriguing open question is whether this approach can provide a justification for reallocation- or group strategy-proofness.

## A Appendix: Proofs

### A.1 Proof of Theorem 1

#### A.1.1 "If"

Let $(\varphi_i, \tau_i)_{i\in N}$ be a fixed-price mechanism with price $p \geq 0$ and lump-sum transfers $(\underline{t}_i)_{i\in N} \in \mathbb{R}^n$. Consider any $c_N \in C^n$, $G \subseteq N$, $\hat{c}_G \in C^{|G|}$ and $(\hat{q}_i, \hat{t}_i)_{i\in G} \in (\mathbb{N} \times \mathbb{R})^{|G|}$ such that

$$\sum_{i\in G} \big[\hat{q}_i - \varphi_i(\hat{c}_G, c_{-G})\big] = 0 = \sum_{i\in G} \big[\hat{t}_i - \tau_i(\hat{c}_G, c_{-G})\big].$$

These two equations and (2) imply that

$$\sum_{i\in G}\hat{t}_i=\sum_{i\in G}\tau_i(\hat{c}_G,c_{-G})=\sum_{i\in G}\big[\varphi_i(\hat{c}_G,c_{-G})p+\underline{t}_i\big]=\sum_{i\in G}(\hat{q}_ip+\underline{t}_i).$$

Moreover, by (1) and (2),

$$\forall i\in G,\quad \hat{q}_ip-c_i(\hat{q}_i)\leq\varphi_i(c_N)p-c_i\big(\varphi_i(c_N)\big)=\tau_i(c_N)-\underline{t}_i-c_i\big(\varphi_i(c_N)\big).$$

It follows that

$$\sum_{i\in G}\big[\hat{t}_i-c_i(\hat{q}_i)\big]=\sum_{i\in G}\big[\hat{q}_ip+\underline{t}_i-c_i(\hat{q}_i)\big]\leq\sum_{i\in G}\Big[\tau_i(c_N)-c_i\big(\varphi_i(c_N)\big)\Big],$$

so $(\varphi_i,\tau_i)_{i\in N}$ is reallocation-proof.

### A.1.2 "Only if"

Let $(\varphi_i,\tau_i)_{i\in N}$ be a reallocation-proof mechanism. Reallocation-proofness implies "bribe-proofness" (Schummer, 2000b): if one agent bribes another to misreport, they do not both become better off. Equivalently, an agent's own loss from misreporting is at least as large as any other agent's gain.[13]

**Definition A.1.** $(\varphi_i,\tau_i)_{i\in N}$ is **bribe-proof** if for all $i,j\in N$, $c_N\in C^n$ and $\hat{c}_i\in C$, we have that $u_i^*(c_N)-u_i(\hat{c}_i,c_{-i}|c_i)\geq u_j^*(\hat{c}_i,c_{-i})-u_j^*(c_N)$.

In a bribe-proof mechanism, each agent's equilibrium utility can only depend on his own report.

**Lemma A.1.** *For all $i\in N$, $c_i\in C$ and $c_{-i},\hat{c}_{-i}\in C^{n-1}$, we have that $u_i^*(c_i,c_{-i})=u_i^*(c_i,\hat{c}_{-i})$.*

*Proof.* Schummer's (2000b) Theorems 2 and 3 establish this independence condition quite generally. His proof requires the set of alternatives to be compact, however, which is not satisfied in our model because quantities can be arbitrarily large. In Supplement B.1.1, we provide an alternative proof tailored to our model. □

Lemma A.1 implies that the set of quantities available to agent $i$, $Q_i\subseteq\mathbb{N}$, as well as their prices, $p_i(q_i)\in\mathbb{R}$ for all $q_i\in Q_i$, must be independent of the

[13] Bribe-proofness only considers collusion between two agents, one of whom misreports. If we relax reallocation-proofness analogously, Theorem 1 still holds.

other agents' reports. Moreover, by strategy-proofness (implied by bribe-proofness), $i$ gets his optimum in $Q_i$.

**Lemma A.2.** *For all $i \in N$, there exists a set $Q_i \subseteq \mathbb{N}$ and an increasing function $p_i \colon \mathbb{N} \to \mathbb{R}$ such that for all $c_N \in C^n$,*

$$\varphi_i(c_N) \in \arg\max_{q_i \in Q_i} \{p_i(q_i) - c_i(q_i)\}, \tag{A.1}$$

$$\tau_i(c_N) = p_i\big(\varphi_i(c_N)\big). \tag{A.2}$$

*Proof.* See Supplement B.1.2. □

Lemma A.2 is as far as we get with bribe-proofness. If we also want to deter collusion among agents who can fully reallocate, then either no agent provides any units or all agents can supply whatever quantity they wish.

**Lemma A.3.** *Either $Q_i = \{0\}$ for all $i \in N$, or $Q_i = \mathbb{N}$ for all $i \in N$.*

*Proof.* There are four steps.

Step 1: For all $i \in N$, $0 \in Q_i$.

By contradiction, suppose there exists $i \in N$ such that $\underline{q}_i := \min\{Q_i\} > 0$. Consider any $j \in N \setminus \{i\}$. Let $c_N \in C^n$ be such that $\varphi_i(c_N) = \underline{q}_i$ and $c_i(\underline{q}_i) > c_j(\varphi_j(c_N) + \underline{q}_i) - c_j(\varphi_j(c_N))$.[14] Both agents gain if $i$ gives his $\underline{q}_i$ units to $j$, accompanied by a transfer $t$ such that $c_i(\underline{q}_i) > t > c_j(\varphi_j(c_N) + \underline{q}_i) - c_j(\varphi_j(c_N))$. This violates reallocation-proofness, so Step 1 holds.

If $Q_i = \{0\}$ for all $i \in N$, we are done. Thus, suppose $Q_i \setminus \{0\} \neq \emptyset$ for some $i \in N$. The three steps below yield that $Q_i = \mathbb{N}$ for all $i \in N$. Since the proofs are similar to Step 1's, we relegate them to Supplement B.1.3.

Step 2: For all $i \in N$, $\sup\{Q_i\} = \infty$.

Step 3: There exists $\Delta q \in \mathbb{N} \setminus \{0\}$ such that for all $i \in N$ and $q_i, \hat{q}_i \in Q_i$ with $q_i < \hat{q}_i$ and $\{q_i, \dots, \hat{q}_i\} \cap Q_i = \{q_i, \hat{q}_i\}$, $\hat{q}_i - q_i = \Delta q$.

Step 4: $\Delta q = 1$. □

---

[14]Such a $c_N$ exists: Pick any $q_j \in Q_j$. By Lemma A.2, there is $c_j \in C$ with $\varphi_j(c_j, \cdot) = q_j$. Let $c_i \in C$ satisfy $c_i(\underline{q}_i) > c_j(q_j + \underline{q}_i) - c_j(q_j)$ and $c_i(q_i) > c_i(\underline{q}_i) + [p_i(q_i) - p_i(\underline{q}_i)]$ for all $q_i \in Q_i \setminus \{\underline{q}_i\}$. Then $i$'s optimum in $Q_i$ is $\underline{q}_i$, so $\varphi_i(c_i, \cdot) = \underline{q}_i$ by Lemma A.2.

The first case of Lemma A.3 ($Q_i = \{0\}$ for all $i \in N$) corresponds to a fixed-price mechanism with $p = 0$. In the second case ($Q_i = \mathbb{N}$ for all $i \in N$), it remains to be shown that the agents' price functions are identical and linear. By contradiction, suppose $p_j(q_j + 1) - p_j(q_j) < p_i(q_i + 1) - p_i(q_i)$ for some $i, j \in N$ and $q_i, q_j \in \mathbb{N}$. Let $c_i \in C$ be very flat until $q_i$, and very steep thereafter. Lemma A.2 implies that $\varphi_i(c_i, \cdot) = q_i$. Similarly, there exists $c_j \in C$ such that $\varphi_j(c_j, \cdot) = q_j$ and, furthermore, $p_j(q_j+1)-p_j(q_j) < c_j(q_j+1) - c_j(q_j) < p_i(q_i + 1) - p_i(q_i)$. In words, agent $j$ would like to provide one additional unit at $i$'s prices but not at his own. Consider any $c_{-i,j} \in C^{n-2}$ and suppose that, instead of being truthful at $(c_i, c_j, c_{-i,j})$, $i$ reports $\hat{c}_i \in C$ such that $\varphi_i(\hat{c}_i, c_j, c_{-i,j}) = q_i + 1$ and then gives the additional unit to $j$, together with a transfer $t$ such that $c_j(q_j+1)-c_j(q_j) < t < p_i(q_i+1)-p_i(q_i)$. This transfer is large enough to cover $j$'s cost increase but small enough to leave some profit to $i$, violating reallocation-proofness. Hence, there must exist $p > 0$ such that $p_i(q_i + 1) - p_i(q_i) = p$ for all $i \in N$ and $q_i \in \mathbb{N}$.

## A.2 Proof of Proposition 1

Let $(\varphi_i, \tau_i)_{i\in N}$ be group strategy-proof (GSP) and individually rational (IR). Consider any $i \in \{1, \dots, n\}$ and define $I := \{1, \dots, i\}$. Let $0^i$ be an $i$-vector of zeros. By induction, we prove that for all $c_{-I} \in C^{n-i}$, there exists $\hat{c}_I \in C^i$ such that for all $c_I \in C^i$ with $c_I > \hat{c}_I$, $\varphi_I(c_I, c_{-I}) = 0^i$. Proposition 1 follows from $i = n$ (in which case $c_{-I}$ is the empty tuple).

The basis ($i = 1$) is a simplified version of the induction step, so we only prove the latter. Consider any $i \in \{2, \dots, n\}$ and suppose the claim above is true for $i - 1$. Fix any $c_{-I} \in C^{n-i}$. Lemma 1 implies that, for each $q_i \in \mathbb{N}$, quantity profile $q_I := (0^{i-1}, q_i)$ is associated with exactly one transfer profile—or none if $q_I$ is not realized at any $c_I \in C^i$. We can thus define a function $\bar{t}_i \colon \mathbb{N} \to \mathbb{R}$ that maps each $(0^{i-1}, q_i)$ into $i$'s transfer:

$$\bar{t}_i(q_i) := \begin{cases} \tau_i(c_I, c_{-I}) & \text{if } \exists c_I \in C^i \text{ s.t. } \varphi_I(c_I, c_{-I}) = (0^{i-1}, q_i) \\ 0 & \text{otherwise.} \end{cases}$$

Let $\hat{c}_i \in C$ be such that $\hat{c}_i(q_i) > \bar{t}_i(q_i)$ for all $q_i \in \mathbb{N} \setminus \{0\}$. By the induction hypothesis, there exists $\hat{c}_{I-i} := (\hat{c}_1, \dots, \hat{c}_{i-1}) \in C^{i-1}$ such that for all $c_{I-i} := (c_1, \dots, c_{i-1}) \in C^{i-1}$ with $c_{I-i} > \hat{c}_{I-i}$, $\varphi_{I-i}(c_{I-i}, \hat{c}_i, c_{-I}) = 0^{i-1}$.

Defining $c_{-i} := (c_{I-i}, c_{-I})$, it follows that $\varphi_I(\hat{c}_i, c_{-i}) = (0^{i-1}, q_i)$ for some $q_i \in \mathbb{N}$. If $q_i > 0$, the construction of $\hat{c}_i$ implies that $i$'s utility is negative. Thus, by IR, $q_i = 0$. In sum,

$$\forall c_{I-i} \in C^{i-1} \text{ s.t. } c_{I-i} > \hat{c}_{I-i}, \quad \varphi_I(\hat{c}_i, c_{-i}) = 0^i. \tag{A.3}$$

It remains to prove that (A.3) still holds when $\hat{c}_i$ is replaced by any $c_i \in C$ with $c_i > \hat{c}_i$. First, we show that $\varphi_i(c_i, c_{-i}) = 0$. If $\varphi_i(c_i, c_{-i}) > 0$, strategy-proofness (SP) is violated for $c_i$ because

$$\begin{aligned} u_i^*(c_i, c_{-i}) &= \tau_i(c_i, c_{-i}) - c_i\big(\varphi_i(c_i, c_{-i})\big) \\ &< \tau_i(c_i, c_{-i}) - \hat{c}_i\big(\varphi_i(c_i, c_{-i})\big) \\ &\leq \tau_i(\hat{c}_i, c_{-i}) - \hat{c}_i\big(\varphi_i(\hat{c}_i, c_{-i})\big) \\ &= \tau_i(\hat{c}_i, c_{-i}) - c_i\big(\varphi_i(\hat{c}_i, c_{-i})\big) = u_i(\hat{c}_i, c_{-i}|c_i). \end{aligned}$$

The strict inequality follows from $\varphi_i(c_i, c_{-i}) > 0$ and $c_i > \hat{c}_i$, the weak inequality from SP for $\hat{c}_i$, and the penultimate equality from $\varphi_i(\hat{c}_i, c_{-i}) = 0$.

Finally, we show that $\varphi_j(c_i, c_{-i}) = 0$ for all $j < i$. Since $\varphi_i(c_i, c_{-i}) = \varphi_i(\hat{c}_i, c_{-i})$, GSP requires that $u_j^*(c_i, c_{-i}) = u_j^*(\hat{c}_i, c_{-i})$. Hence,

$$\forall c_{I-i} \in C^{i-1} \text{ s.t. } c_{I-i} > \hat{c}_{I-i}, \quad u_j^*(c_i, c_{-i}) = u_j^*(\hat{c}_i, c_{-i}). \tag{A.4}$$

Let $\tilde{c}_j \in C$ be such that $\hat{c}_j < \tilde{c}_j < c_j$. Since $j \in I$ and $(\tilde{c}_j, c_{I-i,j}) > \hat{c}_{I-i}$, (A.3) implies that $\varphi_j(\hat{c}_i, \tilde{c}_j, c_{-i,j}) = \varphi_j(\hat{c}_i, c_{-i}) = 0$. Thus, by SP for $\tilde{c}_j$ and $c_j$, $\tau_j(\hat{c}_i, \tilde{c}_j, c_{-i,j}) = \tau_j(\hat{c}_i, c_{-i})$. If $\varphi_j(c_i, c_{-i}) > 0$, then (A.4) is violated at $(\tilde{c}_j, c_{I-i,j})$ because

$$\begin{aligned} u_j^*(\tilde{c}_j, c_{-j}) &\geq \tau_j(c_j, c_{-j}) - \tilde{c}_j\big(\varphi_j(c_j, c_{-j})\big) \\ &> \tau_j(c_j, c_{-j}) - c_j\big(\varphi_j(c_j, c_{-j})\big) = u_j^*(c_j, c_{-j}) \\ &= u_j^*(\hat{c}_i, c_{-i}) = \tau_j(\hat{c}_i, c_{-i}) - c_j(0) \\ &= \tau_j(\hat{c}_i, \tilde{c}_j, c_{-i,j}) - \tilde{c}_j(0) = u_j^*(\hat{c}_i, \tilde{c}_j, c_{-i,j}). \end{aligned}$$

## A.3 Proof of Proposition 2

By contradiction, suppose $(\varphi_i, \tau_i)_{i \in N}$ is group strategy-proof and $q^*$-efficient for some $q^* \in \{1, 2, \dots\}$. By Lemma 1, there exist $t_1^1, t_1^2 \in \mathbb{R}$ such that

$$\forall c_N \in C^n, \quad [\varphi_N(c_N) = (q^*, 0, \dots, 0) \implies \tau_1(c_N) = t_1^1],$$
$$\forall c_N \in C^n, \quad [\varphi_N(c_N) = (0, q^*, 0, \dots, 0) \implies \tau_1(c_N) = t_1^2].$$

For all $c, c' \in C$, we write $c \ll c'$ if $\Delta c(q^*) < \Delta c'(1)$. Let

$$c_1' \ll c_2' \ll c_1 \ll c_2'' \ll c_1'' \ll c_i \quad \forall i \in N \setminus \{1, 2\}. \tag{A.5}$$

Since $c_{-1,2}$ is kept fixed below, we suppress its notation. By $q^*$-efficiency,

$$\varphi_N(c_1, c_2'') = \varphi_N(c_1', c_2') = (q^*, 0, \dots, 0),$$
$$\varphi_N(c_1'', c_2'') = \varphi_N(c_1, c_2') = (0, q^*, 0, \dots, 0).$$

Strategy-proofness for agent 1 requires that

$$u_1^*(c_1, c_2'') \geq u_1(c_1'', c_2''|c_1) \iff t_1^1 - c_1(q^*) \geq t_1^2 - c_1(0),$$
$$u_1^*(c_1, c_2') \geq u_1(c_1', c_2'|c_1) \iff t_1^2 - c_1(0) \geq t_1^1 - c_1(q^*).$$

Since $c_1(0) = 0$, it follows that $c_1(q^*) = t_1^1 - t_1^2$.

Let $\hat{c}_1 \in C$ be a small perturbation of $c_1$ so that $\hat{c}_1(q^*) \neq c_1(q^*)$ and (A.5) still holds when $c_1$ is replaced by $\hat{c}_1$. Repeating the argument above, we get that $\hat{c}_1(q^*) = t_1^1 - t_1^2$ and thus $\hat{c}_1(q^*) = c_1(q^*)$, a contradiction.

## A.4 Proof of Lemma 2

The "if" is obvious. As for the "only if", suppose $(\varphi_i, \tau_i)_{i \in N}$ satisfies no price envy (NPE) and zero transfer for zero quantity (0T0Q). Consider any $c_N \in C^n$. If $\varphi_i(c_N) = 0$ for all $i \in N$, then 0T0Q implies that $\tau_i(c_N) = 0$ for all $i \in N$. Setting $\pi(c_N) := 0$, (3) and (4) hold.

Now suppose $\varphi_j(c_N) > 0$ for some $j \in N$. Consider any $i \in N$ with $\varphi_i(c_N) > 0$. If $\tau_j(c_N)/\varphi_j(c_N) > \tau_i(c_N)/\varphi_i(c_N)$, NPE is violated for $i$ because

$$\varphi_i(c_N) \frac{\tau_j(c_N)}{\varphi_j(c_N)} - c_i\big(\varphi_i(c_N)\big) > \tau_i(c_N) - c_i\big(\varphi_i(c_N)\big) = u_i^*(c_N).$$

Similarly, $\tau_j(c_N)/\varphi_j(c_N) < \tau_i(c_N)/\varphi_i(c_N)$ violates NPE for $j$. Hence, there exists $\pi(c_N) \in \mathbb{R}$ such that for all $i \in N$ with $\varphi_i(c_N) > 0$, $\tau_i(c_N)/\varphi_i(c_N) = \pi(c_N)$ and thus $\tau_i(c_N) = \varphi_i(c_N)\pi(c_N)$. By 0T0Q, $\tau_i(c_N) = \varphi_i(c_N)\pi(c_N)$ also holds if $\varphi_i(c_N) = 0$, establishing (4). (3) is satisfied as well because, by NPE, for all $i \in N$ and $q_i \in \mathbb{N}$,

$$\begin{aligned}\varphi_i(c_N)\pi(c_N) - c_i\big(\varphi_i(c_N)\big) &= \tau_i(c_N) - c_i\big(\varphi_i(c_N)\big) = u_i^*(c_N) \\ &\geq q_i \frac{\tau_j(c_N)}{\varphi_j(c_N)} - c_i(q_i) = q_i\pi(c_N) - c_i(q_i).\end{aligned}$$

## A.5 Proof of Theorem 2

### A.5.1 Part a)

Let $(\varphi_i, \tau_i)_{i\in N}$ be an $n$-price mechanism with $0 = p(0) \leq p(1) \leq \cdots \leq p(n) < \infty$. Anonymity and uniform pricing obviously hold, so we only show that $(\varphi_i, \tau_i)_{i\in N}$ is group strategy-proof. Consider any $c_N \in C^n$, $G \subseteq N$ and $\hat{c}_G \in C^{|G|}$. Let $\hat{c}_i \coloneqq c_i$ for all $i \in N \setminus G$.

For some agent in $G$ to gain from misreport $\hat{c}_G$, the price must increase, that is, $\widehat{m} \coloneqq \mu(\hat{c}_N) > \mu(c_N)$. Using the definition of $\mu$, it is easy to see that

$$\big|\{i \in N : \hat{c}_i(1) \leq p(\widehat{m})\}\big| = \widehat{m} > \big|\{i \in N : c_i(1) \leq p(\widehat{m})\}\big|.$$

Since $\hat{c}_i = c_i$ for all $i \in N \setminus G$, there must exist $i \in G$ with $\hat{c}_i(1) \leq p(\widehat{m}) < c_i(1)$. By (5), $\varphi_i(\hat{c}_N) > 0 = \varphi_i(c_N)$ and thus $u_i(\hat{c}_N|c_i) < u_i^*(c_N)$.

### A.5.2 Part b)

Let $(\varphi_i, \tau_i)_{i\in N}$ be a group-strategy-proof (GSP), anonymous uniform-price mechanism. Note that, by Definition 9, $(\varphi_i, \tau_i)_{i\in N}$ is individually rational (IR). Definition 9 also implies "equal treatment of equals", that is, agents with identical cost functions get the same equilibrium payoffs. Nonetheless, their allocations may differ. Lemma A.4 below deals with this technicality. Recall that $c_G \in C_*^{|G|}$ means $c_i = c_j$ for all $i, j \in G$.

**Lemma A.4.** *For all $G \subseteq N$, $(c_G, c_{-G}) \in C_*^{|G|} \times C^{n-|G|}$ and $i \in G$, there exists $\hat{c}_G \in C_*^{|G|}$ such that for all $j \in G$, $\alpha_j(\hat{c}_G, c_{-G}) = \alpha_i(c_G, c_{-G})$.*

*Proof.* Since the proof is tedious, we relegate it to Supplement B.3. □

Our next lemma is a variant of the "taxation principle" (Rochet, 1985), applied to groups of agents with identical costs. They collectively face a price function, which is independent of their own reports, and then choose their optimal quantity. This result hinges on group strategy-proofness and anonymity; uniform pricing enters only indirectly via Lemma A.4.

**Lemma A.5.** *For all $g \in \{1, \dots, n\}$, there exists a function $\pi_*^g \colon \mathbb{N} \times C^{n-g} \to \mathbb{R}_+$ with the following two properties:*

*(i) For all $q \in \mathbb{N}$, $\pi_*^g(q, \cdot)$ is symmetric*[15]*.*

*(ii) For all $G \subseteq N$ with $|G| = g$, $(c_G, c_{-G}) \in C_*^g \times C^{n-g}$ and $i \in G$,*

$$\varphi_i(c_N) \in \arg\max_{q \in \mathbb{N}} \{\pi_*^g(q, c_{-G}) - c_i(q)\}, \tag{A.6}$$

$$\tau_i(c_N) = \pi_*^g(\varphi_i(c_N), c_{-G}). \tag{A.7}$$

*Proof.* Consider any $G \subseteq N$ and $c_{-G} \in C^{n-g}$, where $g := |G|$. We establish the existence of a function $\pi_G^*(\cdot, c_{-G}) \colon \mathbb{N} \to \mathbb{R}_+$ such that for all $c_G \in C_*^g$ and $i \in G$,

$$\varphi_i(c_N) \in \arg\max_{q \in \mathbb{N}} \{\pi_G^*(q, c_{-G}) - c_i(q)\}, \tag{A.8}$$

$$\tau_i(c_N) = \pi_G^*(\varphi_i(c_N), c_{-G}). \tag{A.9}$$

Anonymity implies that $\pi_G^*$ is symmetric in $c_{-G}$ and identical for all groups $G$ of size $g$, so we can define $\pi_*^g := \pi_G^*$.

We start by showing that for all $c_G, \tilde{c}_G \in C_*^g$ and $i \in G$,

$$\varphi_i(c_G, c_{-G}) \geq \varphi_i(\tilde{c}_G, c_{-G}) \implies \tau_i(c_G, c_{-G}) \geq \tau_i(\tilde{c}_G, c_{-G}). \tag{A.10}$$

Since $c_G \in C_*^g$, $u_j^*(c_G, c_{-G}) = u_i^*(c_G, c_{-G})$ for all $j \in G$. By Lemma A.4, there exists $\hat{c}_G \in C_*^g$ such that $\varphi_j(\hat{c}_G, c_{-G}) = \varphi_i(\tilde{c}_G, c_{-G})$ and $\tau_j(\hat{c}_G, c_{-G}) = \tau_i(\tilde{c}_G, c_{-G})$. Contrary to (A.10), suppose $\varphi_i(c_G, c_{-G}) \geq \varphi_i(\tilde{c}_G, c_{-G})$ and $\tau_i(c_G, c_{-G}) < \tau_i(\tilde{c}_G, c_{-G})$. Then GSP is violated because for all $j \in G$,

$$\begin{aligned} u_j^*(c_G, c_{-G}) &= u_i^*(c_G, c_{-G}) \\ &= \tau_i(c_G, c_{-G}) - c_i\big(\varphi_i(c_G, c_{-G})\big) \end{aligned}$$

[15] A function is symmetric if its value is unchanged by any permutation of its arguments.

$$< \tau_i(\tilde{c}_G, c_{-G}) - c_i\big(\varphi_i(\tilde{c}_G, c_{-G})\big)$$
$$= \tau_j(\hat{c}_G, c_{-G}) - c_j\big(\varphi_j(\hat{c}_G, c_{-G})\big) = u_j(\hat{c}_G, c_{-G}|c_j).$$

(A.10) implies that, across cost profiles where the agents in $G$ make identical reports (i.e. $c_G \in C_*^g$), the transfer to each agent $i \in G$ is entirely determined by his quantity. Moreover, by Lemma A.4, the range of quantities and transfers is the same for all agents in $G$. Hence, we can define a function $\pi_G^*(\cdot, c_{-G})\colon \mathbb{N} \to \mathbb{R}$ that maps each $q \in \mathbb{N}$ into $i$'s transfer:

$$\pi_G^*(q, c_{-G}) \coloneqq \begin{cases} \tau_i(c_G, c_{-G}) & \text{if } \exists c_G \in C_*^g \text{ s.t. } \varphi_i(c_G, c_{-G}) = q \\ 0 & \text{otherwise.} \end{cases} \tag{A.11}$$

Let $c_G^\circ \in C_*^g$ be such that for all $i \in G$ and $q \in \mathbb{N}\backslash\{0\}$, $c_i^\circ(q) > \pi_G^*(q, c_{-G})$. By IR, $\varphi_i(c_G^\circ, c_{-G}) = 0$ and $\tau_i(c_G^\circ, c_{-G}) = \pi_G^*(0, c_{-G}) \geq 0$. From (A.10), it follows that $\pi_G^*(q, c_{-G}) \geq 0$ for all $q \in \mathbb{N}$.

Note that (A.9) holds by construction. To prove (A.8), consider any $c_G \in C_*^g$ and $i \in G$. Defining $q \coloneqq \varphi_i(c_G, c_{-G})$, we wish to show that $\pi_G^*(q, c_{-G}) - c_i(q) \geq \pi_G^*(\tilde{q}, c_{-G}) - c_i(\tilde{q})$ for all $\tilde{q} \in \mathbb{N}$. There are two cases:

Case 1: $\exists \tilde{c}_G \in C_*^g$ s.t. $\varphi_i(\tilde{c}_G, c_{-G}) = \tilde{q}$.

(A.11) implies that $\tau_i(\tilde{c}_G, c_{-G}) = \pi_G^*(\tilde{q}, c_{-G})$. By Lemma A.4, there exists $\hat{c}_G \in C_*^g$ such that for all $j \in G$, $\varphi_j(\hat{c}_G, c_{-G}) = \tilde{q}$ and $\tau_j(\hat{c}_G, c_{-G}) = \pi_G^*(\tilde{q}, c_{-G})$. Since $c_j = c_i$, it follows that $u_j(\hat{c}_G, c_{-G}|c_j) = u_i(\hat{c}_G, c_{-G}|c_i)$ and $u_j^*(c_G, c_{-G}) = u_i^*(c_G, c_{-G})$. By GSP, $u_i^*(c_G, c_{-G}) \geq u_i(\hat{c}_G, c_{-G}|c_i)$. Thus,

$$\pi_G^*(q, c_{-G}) - c_i(q) = \tau_i(c_G, c_{-G}) - c_i\big(\varphi_i(c_G, c_{-G})\big) = u_i^*(c_G, c_{-G})$$
$$\geq u_i(\hat{c}_G, c_{-G}|c_i) = \tau_i(\hat{c}_G, c_{-G}) - c_i\big(\varphi_i(\hat{c}_G, c_{-G})\big) = \pi_G^*(\tilde{q}, c_{-G}) - c_i(\tilde{q}).$$

Case 2: $\nexists \tilde{c}_G \in C_*^g$ s.t. $\varphi_i(\tilde{c}_G, c_{-G}) = \tilde{q}$.

By (A.11), $\pi_G^*(\tilde{q}, c_{-G}) = 0$. Recall from above that $\varphi_i(c_G^\circ, c_{-G}) = 0$ and $\pi_G^*(0, c_{-G}) \geq 0$. Hence, by Case 1 applied to $c_G^\circ \in C_*^g$,

$$\pi_G^*(q, c_{-G}) - c_i(q) \geq \pi_G^*(0, c_{-G}) - c_i(0) \geq 0 > \pi_G^*(\tilde{q}, c_{-G}) - c_i(\tilde{q}). \qquad \square$$

Exploiting the uniform-price requirement, we now show that the functions $\{\pi_*^1, \ldots, \pi_*^n\}$ from Lemma A.5 are all linear in $q$. Define a reference function $\pi^0_*$ by $\pi^0_*(q, c_N) \coloneqq q\pi(c_N)$ for all $q \in \mathbb{N}$ and $c_N \in C^n$, where $\pi(c_N)$ is the uniform price from Definition 9. For all $g \in \{0, \dots, n\}$, $G \subseteq N$ with $|G| = g$, and $c_{-G} \in C^{n-g}$, the set of allocations associated with $\pi^g_*(\cdot, c_{-G})$ is

$$A^g(c_{-G}) \coloneqq \Big\{\big(q, \pi^g_*(q, c_{-G})\big) : q \in \mathbb{N}\Big\}. \tag{A.12}$$

By definition, $A^0(c_N)$ is the graph of a linear function with slope $\pi(c_N)$. Lemma A.7 below extends the linearity inductively to $A^1$, $A^2$, etc. The proof uses the following auxiliary result.

**Lemma A.6.** *For all $g \in \{1, \dots, n\}$, $G \subseteq N$ with $|G| = g$, $c_{-G} \in C^{n-g}$, $i \in G$ and $c_i \in C$,*

$$\mathrm{Opt}\big(c_i, A^g(c_{-G})\big) \cap \mathrm{Opt}\big(c_i, A^{g-1}(c_i, c_{-G})\big) \neq \emptyset. \tag{A.13}$$

*Proof.* Consider any $g \in \{1, \dots, n\}$, $G \subseteq N$ with $|G| = g$, $c_{-G} \in C^{n-g}$, $i \in G$ and $c_i \in C$. Let $c_j = c_i$ for all $j \in G \setminus \{i\}$, so $c_G \in C^g_*$. By Lemma A.5, $\alpha_i(c_N) \in \mathrm{Opt}(c_i, A^g(c_{-G}))$. Definition 9 says that $\alpha_i(c_N) \in \mathrm{Opt}(c_i, A^0(c_N))$, so (A.13) holds for $g = 1$. If $g \in \{2, \dots, n\}$, take any $j \in G \setminus \{i\}$. Since $c_{G-j} \in C^{g-1}_*$, Lemma A.5 implies that $\alpha_i(c_N) \in \mathrm{Opt}(c_i, A^{g-1}(c_j, c_{-G}))$. Combined with $\alpha_i(c_N) \in \mathrm{Opt}(c_i, A^g(c_{-G}))$ and $c_j = c_i$, we get (A.13). □

**Lemma A.7.** *For all $g \in \{1, \dots, n\}$, $G \subseteq N$ with $|G| = g$, $c_{-G} \in C^{n-g}$ and $q \in \mathbb{N}$, we have that $\pi^g_*(q, c_{-G}) = q\pi^g_*(1, c_{-G})$.*

*Proof.* By definition of $\pi^0_*$, the claim holds for $g = 0$. Proceeding inductively, let $g \in \{1, \dots, n\}$ and suppose the claim is true for $g - 1$. Consider any $G \subseteq N$ with $|G| = g$, and any $c_{-G} \in C^{n-g}$. By Lemma A.5, $\pi^g_*(q, c_{-G}) \geq 0$ for all $q \in \mathbb{N}$. Suppose there exists $q \in \mathbb{N}$ such that $\pi^g_*(q, c_{-G}) > 0$; otherwise, we are done. The proof has three steps:

1. $\pi^g_*(0, c_{-G}) = 0$.
2. $\pi^g_*(1, c_{-G}) > 0$.
3. For all $\hat{q} \in \{1, 2, \dots\}$, if $\pi^g_*(q, c_{-G}) = q\pi^g_*(1, c_{-G})$ for all $q \in \{0, \dots, \hat{q}\}$, then $\pi^g_*(\hat{q} + 1, c_{-G}) = (\hat{q} + 1)\pi^g_*(1, c_{-G})$.

The argument behind each step is similar, so we only establish 2, assuming that 1 holds. Define $\underline{q} \coloneqq \min\{q \in \mathbb{N} : \pi^g_*(q, c_{-G}) > 0\}$. By contradiction, suppose that $\underline{q} > 1$. Let $c_G \in C_*^g$ be such that for all $i \in G$,

$$\frac{\pi_*^g(\underline{q}, c_{-G})}{\underline{q}} - c_i(1) > 2\frac{\pi_*^g(\underline{q}, c_{-G})}{\underline{q}} - c_i(2) > \cdots > \pi_*^g(\underline{q}, c_{-G}) - c_i(\underline{q}) > 0$$

and $0 \geq \pi_*^g(q, c_{-G}) - c_i(q)$ for all $q \in \{\underline{q}+1, \underline{q}+2, \ldots\}$; see Figure A.1. By construction,

$$\text{Opt}\big(c_i, A^g(c_{-G})\big) = \Big\{\big(\underline{q}, \pi_*^g(\underline{q}, c_{-G})\big)\Big\}.$$

From (A.13), it follows that

$$\big(\underline{q}, \pi_*^g(\underline{q}, c_{-G})\big) \in \text{Opt}\big(c_i, A^{g-1}(c_i, c_{-G})\big). \tag{A.14}$$

Since $\underline{q} > 0$, the induction hypothesis implies that

$$A^{g-1}(c_i, c_{-G}) = \left\{\left(q, q\frac{\pi_*^g(\underline{q}, c_{-G})}{\underline{q}}\right) : q \in \mathbb{N}\right\}.$$

Hence, by construction of $c_i$,

$$\text{Opt}\big(c_i, A^{g-1}(c_i, c_{-G})\big) = \left\{\left(1, \frac{\pi_*^g(\underline{q}, c_{-G})}{\underline{q}}\right)\right\}. \tag{A.15}$$

Since $\underline{q} \neq 1$, (A.15) contradicts (A.14). Thus, $\pi_*^g(1, c_{-G}) > 0$. □

To simplify notation, define $\pi^g(c_{-G}) := \pi_*^g(1, c_{-G})$ for all $g \in \{1, \ldots, n\}$, $G \subseteq N$ with $|G| = g$, and $c_{-G} \in C^{n-g}$. By (A.12) and Lemma A.7,

$$A^g(c_{-G}) = \Big\{\big(q, q\pi^g(c_{-G})\big) : q \in \mathbb{N}\Big\}. \tag{A.16}$$

The next lemma relates $\pi^g$ and $\pi^{g-1}$, showing that the "removal" of an agent from a symmetric group cannot raise the price for the remaining group members.

**Lemma A.8.** *For all $g \in \{2, \ldots, n\}$, $G \subseteq N$ with $|G| = g$, $c_{-G} \in C^{n-g}$, $i \in G$ and $c_i \in C$,*

$$\begin{aligned} \pi^{g-1}(c_i, c_{-G}) &= \pi^g(c_{-G}) \quad \textit{if } c_i(1) \leq \pi^g(c_{-G}), \\ \pi^{g-1}(c_i, c_{-G}) &\leq \pi^g(c_{-G}) \quad \textit{if } c_i(1) > \pi^g(c_{-G}). \end{aligned}$$

*Proof.* Consider any $g \in \{2, \ldots, n\}$, $G \subseteq N$ with $|G| = g$, $c_{-G} \in C^{n-g}$ and

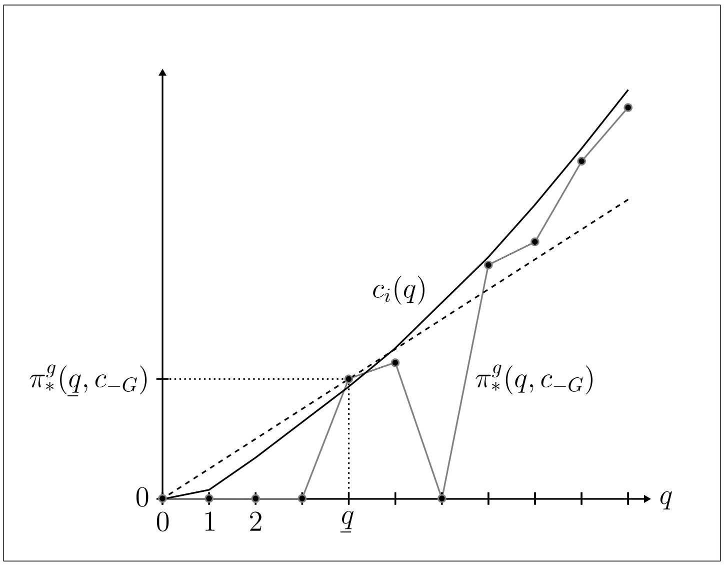


**Figure A.1:** Illustration of $c_i$ in the proof of Lemma A.7

$i \in G$. There are three steps:

Step 1: $\forall c_i \in C$, $\pi^{g-1}(c_i, c_{-G}) \leq \pi^g(c_{-G})$.

To the contrary, suppose there exists $\hat{c}_i \in C$ with $\pi^g(c_{-G}) < \pi^{g-1}(\hat{c}_i, c_{-G})$. Then, by (A.16), $A^g(c_{-G}) \cap A^{g-1}(\hat{c}_i, c_{-G}) = \{(0,0)\}$. Lemma A.6 implies that $(0,0) \in \text{Opt}(\hat{c}_i, A^{g-1}(\hat{c}_i, c_{-G}))$, so $\pi^{g-1}(\hat{c}_i, c_{-G}) \leq \hat{c}_i(1)$. Let $c_G \in C^g_*$ be such that for all $j \in G$, $\pi^g(c_{-G}) < c_j(1) < \pi^{g-1}(\hat{c}_i, c_{-G})$ and $c_j \leq \hat{c}_i$. For all $j \in G$, $\pi^g(c_{-G}) < c_j(1)$ implies that $\text{Opt}(c_j, A^g(c_{-G})) = \{(0,0)\}$ and thus, by Lemma A.5, $\alpha_j(c_N) = (0,0)$. Also, for all $j \in G \setminus \{i\}$, $c_j(1) < \pi^{g-1}(\hat{c}_i, c_{-G})$ implies that $u^*_j(\hat{c}_i, c_{-i}) \geq \pi^{g-1}(\hat{c}_i, c_{-G}) - c_j(1) > 0 = u^*_j(c_N)$. Finally, using $c_i \leq \hat{c}_i$ and IR, $u_i(\hat{c}_i, c_{-i}|c_i) \geq u^*_i(\hat{c}_i, c_{-i}) \geq 0 = u^*_i(c_N)$. Therefore, GSP is violated for group $G$ at $c_N$ via $(\hat{c}_i, c_{-i})$.

Step 2: $\forall c_i \in C$ s.t. $c_i(1) < \pi^g(c_{-G})$, $\pi^{g-1}(c_i, c_{-G}) = \pi^g(c_{-G})$.

For all $c_i \in C$ such that $c_i(1) < \pi^g(c_{-G})$, $(0,0) \notin \text{Opt}(c_i, A^g(c_{-G}))$. Hence, by Lemma A.6, $A^g(c_{-G})$ and $A^{g-1}(c_i, c_{-G})$ share an allocation with positive quantity. This is only possible if $\pi^g(c_{-G}) = \pi^{g-1}(c_i, c_{-G})$.

Step 3: $\forall c_i \in C$ s.t. $c_i(1) = \pi^g(c_{-G})$, $\pi^{g-1}(c_i, c_{-G}) = \pi^g(c_{-G})$.

By contradiction, suppose there exists $c_i \in C$ such that $c_i(1) = \pi^g(c_{-G})$ and $\pi^{g-1}(c_i, c_{-G}) \neq \pi^g(c_{-G})$. By Step 1, $\pi^{g-1}(c_i, c_{-G}) < \pi^g(c_{-G})$. For all $\hat{c}_i \in C$

such that $\hat{c}_i(1) < \pi^g(c_{-G}) < \Delta\hat{c}_i(2)$, Step 2 states that $\pi^{g-1}(\hat{c}_i, c_{-G}) = \pi^g(c_{-G})$. Let $c_{G-i} \in C_*^{g-1}$ be such that for all $j \in G \setminus \{i\}$, $\pi^{g-1}(c_i, c_{-G}) < c_j(1) < \pi^{g-1}(\hat{c}_i, c_{-G})$. By Lemmas A.5 and A.7, $u_j^*(c_N) = 0 < \pi^{g-1}(\hat{c}_i, c_{-G}) - c_j(1) \leq u_j^*(\hat{c}_i, c_{-i})$. We now show that $u_i^*(c_N) = u_i(\hat{c}_i, c_{-i}|c_i)$, so GSP is violated. Since $c_j(1) < \pi^g(c_{-G})$, Step 2 implies that $\pi^{g-1}(c_j, c_{-G}) = \pi^g(c_{-G})$. Hence, for all $k \in G\setminus\{i,j\}$, $c_k(1) < \pi^{g-1}(c_j, c_{-G})$. Step 2 holds for groups of any size, so $\pi^{g-2}(c_j, c_k, c_{-G}) = \pi^{g-1}(c_j, c_{-G})$. Repeating this argument for all agents in $G \setminus \{i\}$, we obtain that $\pi^1(c_{-i}) = \pi^g(c_{-G})$. Since $c_i(1) = \pi^g(c_{-G})$, $\text{Opt}(c_i, A^1(c_{-i})) = \{(0,0), (1, \pi^g(c_{-G}))\}$ and thus $u_i^*(c_N) = 0$. Similarly, $\hat{c}_i(1) < \pi^g(c_{-G}) < \Delta\hat{c}_i(2)$ implies that $\alpha_i(\hat{c}_i, c_{-i}) = (1, \pi^g(c_{-G}))$ and thus $u_i(\hat{c}_i, c_{-i}|c_i) = 0$. □

The next lemma pins down the individual unit price $\pi^1(c_{-i})$ that agent $i \in N$ faces when the other agents report $c_{-i} \in C^{n-1}$. Loosely speaking, the proof starts with the price $\pi^n$ that $i$ would face if all other agents had the same cost as him. Then we successively remove everyone except for $i$ from the symmetric group. By Lemma A.8, the price for the remaining group members remains unchanged or decreases, depending on the first-unit cost of the agent who is being removed.

**Lemma A.9.** *There exists a sequence of prices $0 \leq p(1) \leq \cdots \leq p(n) < \infty$ such that for all $i \in N$ and $c_{-i} \in C^{n-1}$,*

$$\pi^1(c_{-i}) = p\big(\mu^1(c_{-i})\big),$$

*where*

$$\mu^1(c_{-i}) := \max\Big\{m \in \{1, \ldots, n\} : 1 + \big|\{j \in N \setminus \{i\} : c_j(1) \leq p(m)\}\big| = m\Big\}. \tag{A.17}$$

*Proof.* By induction, consider any $f \in \{1, \ldots, n-1\}$ and suppose there exists a sequence of prices $0 \leq p(f+1) \leq \cdots \leq p(n) < \infty$ such that for all $g \in \{f+1, \ldots, n\}$, $G \subseteq N$ with $|G| = g$, and $c_{-G} \in C^{n-g}$,

$$\pi^g(c_{-G}) = p\big(\mu^g(c_{-G})\big), \tag{A.18}$$

where

$$\mu^g(c_{-G}) := \max\Big\{m \in \{g, \ldots, n\} : g + \big|\{j \in N \setminus G : c_j(1) \leq p(m)\}\big| = m\Big\}. \tag{A.19}$$

We show that there exists $p(f) \in [0, p(f+1)]$ such that (A.18) and (A.19) hold for $g = f$.

As for the basis, note that $c_{-G}$ is the empty tuple when $G = N$, so $\pi^n$ is constant. Defining $p(n) := \pi^n$, (A.18) and (A.19) hold trivially for $g = n$.

Now consider any $F \subseteq N$ with $|F| = f$, and any $c_{-F} \in C^{n-f}$. Without loss of generality, let us relabel the agents so that $F = \{1, \ldots, f\}$ and $c_{f+1}(1) \leq \cdots \leq c_n(1)$. Define $F1 := F \cup \{f+1\}$. Depending on the cost of agent $f+1$, we distinguish two cases.

Case 1: Suppose $c_{f+1}(1) > \pi^{f+1}(c_{-F1})$.

First, we show that $c_g(1) > p(g)$ for all $g \in \{f+1, \ldots, n\}$. By contradiction, suppose that $h := \max\{g \in \{f+1, \ldots, n\} : c_g(1) \leq p(g)\}$ exists. Then $c_{f+1}(1) \leq \cdots \leq c_h(1) \leq p(h) \leq p(h+1) < c_{h+1}(1) \leq \cdots \leq c_n(1)$, so

$$f + 1 + \big|\{j \in \{f+2, \ldots, n\} : c_j(1) \leq p(h)\}\big| = f + 1 + |\{f+2, \ldots, h\}| = h.$$

From (A.19), it follows that $\mu^{f+1}(c_{-F1}) \geq h$. Combined with (A.18), we get that $\pi^{f+1}(c_{-F1}) = p(\mu^{f+1}(c_{-F1})) \geq p(h) \geq c_h(1) \geq c_{f+1}(1) > \pi^{f+1}(c_{-F1})$, a contradiction.

Second, we show that $\mu^g(c_{-G}) = g$ for all $g \in \{f+1, \ldots, n\}$ and $G := \{1, \ldots, g\}$. Since $p(g) < c_g(1) \leq c_{g+1}(1) \leq \cdots \leq c_n(1)$,

$$g + \big|\{j \in \{g+1, \ldots, n\} : c_j(1) \leq p(g)\}\big| = g. \tag{A.20}$$

Similarly, for all $h \in \{g+1, \ldots, n\}$, $p(h) < c_h(1) \leq c_{h+1}(1) \leq \cdots \leq c_n(1)$. Thus,

$$\begin{aligned} &g + \big|\{j \in \{g+1, \ldots, n\} : c_j(1) \leq p(h)\}\big| \\ &\leq g + \big|\{g+1, \ldots, h-1\}\big| = h - 1 < h. \end{aligned} \tag{A.21}$$

(A.20) and (A.21) together imply that $\mu^g(c_{-G}) = g$.

Third, we show that $\alpha_g(\cdot, c_{-F}) = (0, 0)$ for all $g \in \{f+1, \ldots, n\}$. Define $G := \{1, \ldots, g\}$. For all $c_F \in C^f$, Lemma A.8 implies that $\pi^1(c_{-g}) \leq \pi^g(c_{-G})$. Moreover, by (A.18) and the previous two paragraphs, $\pi^g(c_{-G}) = p(\mu^g(c_{-G})) = p(g) < c_g(1)$. Thus, $\pi^1(c_{-g}) < c_g(1)$. From Lemmas A.5 and A.7, it follows that $\alpha_g(c_F, c_{-F}) = (0, 0)$.

Fourth, we show that $\pi^f(\hat{c}_{-F}) = \pi^f(c_{-F})$ for all $\hat{c}_{-F} \in C^{n-f}$ to which

Case 1 applies (i.e. $\hat{c}_{f+1}(1) > \pi^{f+1}(\hat{c}_{-F1})$, once the agents have been relabeled as above). Repeating the arguments from the previous three paragraphs, we get that $\alpha_g(\cdot, \hat{c}_{-F}) = (0,0) = \alpha_g(\cdot, c_{-F})$. If $\pi^f(\hat{c}_{-F}) < \pi^f(c_{-F})$, there exists $c_F \in C_*^f$ such that $u_i^*(c_F, \hat{c}_{-F}) < u_i^*(c_F, c_{-F})$ for all $i \in F$, violating GSP. An analogous contradiction occurs if $\pi^f(\hat{c}_{-F}) > \pi^f(c_{-F})$. Hence, $\pi^f(\hat{c}_{-F}) = \pi^f(c_{-F})$.

It follows that there exists $p(f) \geq 0$ such that $\pi^f(\hat{c}_{-F}) = p(f)$ for all $\hat{c}_{-F} \in C^{n-f}$ that fall into Case 1. Note that $p(f) \leq p(f+1)$ because $p(f) = \pi^f(c_{-F}) \leq \pi^{f+1}(c_{-F1}) = p(\mu^{f+1}(c_{-F1})) = p(f+1)$ by Lemma A.8, (A.18) and the results above.

Finally, we show that $\mu^f(c_{-F}) = f$, so (A.18) holds for $g = f$. For all $g \in \{f+1, \ldots, n\}$, we know that $p(f) \leq p(g) < c_g(1)$. Thus,

$$f + \big|\{j \in \{f+1, \ldots, n\} : c_j(1) \leq p(f)\}\big| = f$$

and

$$\begin{aligned} & f + \big|\{j \in \{f+1, \ldots, n\} : c_j(1) \leq p(g)\}\big| \\ \leq\ & f + |\{f+1, \ldots, g-1\}| = g - 1 < g. \end{aligned}$$

From (A.19), it follows that $\mu^f(c_{-F}) = f$.

Case 2: Suppose $c_{f+1}(1) \leq \pi^{f+1}(c_{-F1})$.

By Lemma A.8, $c_{f+1}(1) \leq \pi^{f+1}(c_{-F1})$ implies that $\pi^f(c_{-F}) = \pi^{f+1}(c_{-F1})$. Moreover, by (A.18), $\pi^{f+1}(c_{-F1}) = p(\mu^{f+1}(c_{-F1}))$. Using arguments similar to those from Case 1, it can be shown that $\mu^{f+1}(c_{-F1}) = \mu^f(c_{-F})$. Thus, $\pi^f(c_{-F}) = p(\mu^f(c_{-F}))$, as desired. □

Lemmas A.5, A.7 and A.9 imply that for all $c_N \in C^n$ and $i \in N$,

$$\begin{aligned} \varphi_i(c_N) &\in \arg\max_{q_i \in \mathbb{N}} \Big\{ q_i p\big(\mu^1(c_{-i})\big) - c_i(q_i) \Big\}, \\ \tau_i(c_N) &= \varphi_i(c_N) p\big(\mu^1(c_{-i})\big). \end{aligned}$$

Define $p(0) := 0$ and for all $c_N \in C^n$,

$$\mu(c_N) := \max\Big\{ m \in \{0, \ldots, n\} : \big|\{j \in N : c_j(1) \leq p(m)\}\big| = m \Big\}. \quad \text{(A.22)}$$

Our final lemma shows that agent $i$'s set of optima is the same at his individual price $\pi^1(c_{-i}) = p(\mu^1(c_{-i}))$ and at the uniform price $\pi(c_N) := p(\mu(c_N))$.

**Lemma A.10.** *For all* $c_N \in C^n$ *and* $i \in N$,

$$\arg\max_{q_i \in \mathbb{N}} \Big\{ q_i p\big(\mu^1(c_{-i})\big) - c_i(q_i) \Big\} = \arg\max_{q_i \in \mathbb{N}} \Big\{ q_i p\big(\mu(c_N)\big) - c_i(q_i) \Big\}.$$

*Proof.* Consider any $c_N \in C^n$ and $i \in N$. First, suppose $c_i(1) \le p(\mu(c_N))$. By (A.22),

$$\begin{aligned} \mu(c_N) &= \Big| \Big\{ j \in N : c_j(1) \le p\big(\mu(c_N)\big) \Big\} \Big| \\ &= 1 + \Big| \Big\{ j \in N \setminus \{i\} : c_j(1) \le p\big(\mu(c_N)\big) \Big\} \Big|. \end{aligned}$$

From (A.17), it follows that $\mu^1(c_{-i}) \ge \mu(c_N)$ and thus $p(\mu^1(c_{-i})) \ge p(\mu(c_N))$. Since now $c_i(1) \le p(\mu^1(c_{-i}))$, we can apply (A.17) and (A.22) again to obtain that $\mu(c_N) \ge \mu^1(c_{-i})$. Therefore, $\mu(c_N) = \mu^1(c_{-i})$.

Second, suppose $c_i(1) > p(\mu(c_N))$. By above, $c_i(1) \le p(\mu^1(c_{-i}))$ would imply that $\mu^1(c_{-i}) \le \mu(c_N)$ and thus $p(\mu^1(c_{-i})) \le p(\mu(c_N))$. But then $c_i(1) \le p(\mu(c_N))$, a contradiction. It follows that $c_i(1) > p(\mu^1(c_{-i}))$, so

$$\arg\max_{q_i \in \mathbb{N}} \Big\{ q_i p\big(\mu^1(c_{-i})\big) - c_i(q_i) \Big\} = \{0\} = \arg\max_{q_i \in \mathbb{N}} \Big\{ q_i p\big(\mu(c_N)\big) - c_i(q_i) \Big\}. \quad \square$$

Finally, the agents' equilibrium payoffs do not change when—as stated in (5)—we pick the largest optimal quantity in cases of indifference.

## A.6 Proof of Proposition 3

Suppose the agents' costs are independent and identically distributed.

First, consider a fixed-price mechanism with price $p \ge 0$. Breaking indifferences in favor of the largest quantity, an agent with cost function $c \in C$ optimally supplies $q^*(p, c) := \max\{\arg\max_{q \in \mathbb{N}}\{qp - c(q)\}\}$. The agent's contribution to the principal's expected value is $V(p) := (v - p)\mathbb{E}[q^*(p, c)]$, where the expectation is taken over $c \in C$.

Next, consider an $n$-price mechanism with prices $0 \le p(1) \le \ldots \le p(n) < \infty$. By Lemma A.10, the principal's expected value due to an individual agent $i \in N$ can be calculated as follows:

$$V_n\big(p(1), \ldots, p(n)\big) := \sum_{m=1}^{n} \mathbb{P}\big[\mu^1(c_{-i}) = m\big] V\big(p(m)\big).$$

The principal's total expected value is $nV_n\big(p(1),\dots,p(n)\big)$. By Lemma A.9,

$$\mathbb{P}\big[\mu^1(c_{-i})=n\big]=\mathbb{P}\big[\forall j\in N\setminus\{i\},\ c_j(1)\le p(n)\big]=\mathbb{P}\big[c(1)\le p(n)\big]^{n-1},$$
$$\mathbb{P}\big[\mu^1(c_{-i})=n-1\big]=(n-1)\mathbb{P}\big[c(1)>p(n)\big]\mathbb{P}\big[c(1)\le p(n-1)\big]^{n-2},$$

and so on. In general, for all $m\in\{2,\dots,n\}$, $\mathbb{P}\big[\mu^1(c_{-i})=m\big]$ does not depend on $p(m-1),\dots,p(1)$. Moreover,

$$\mathbb{P}\big[\mu^1(c_{-i})=1\big]=1-\sum_{m=2}^{n}\mathbb{P}\big[\mu^1(c_{-i})=m\big].$$

Hence, $p(1)$ enters $V_n\big(p(1),\dots,p(n)\big)$ only through $V\big(p(1)\big)$:

$$\begin{aligned}&V_n\big(p(1),\dots,p(n)\big)\\&=\underbrace{\sum_{m=2}^{n}\mathbb{P}\big[\mu^1(c_{-i})=m\big]V\big(p(m)\big)+\left(1-\sum_{m=2}^{n}\mathbb{P}\big[\mu^1(c_{-i})=m\big]\right)}_{\text{independent of }p(1)}V\big(p(1)\big).\end{aligned}\tag{A.23}$$

Let us replace $p(1)$ with $p_1\in\big\{p(1),p(2)\big\}$ such that $V(p_1)=\max\big\{V\big(p(1)\big),V\big(p(2)\big)\big\}$. By (A.23),

$$\begin{aligned}&V_n\big(p(1),\dots,p(n)\big)\\&\le V_n\big(p_1,p(2),\dots,p(n)\big)\\&=\underbrace{V(p_1)+\sum_{m=3}^{n}\mathbb{P}\big[\mu^1(c_{-i})=m\big]\big[V\big(p(m)\big)-V(p_1)\big]}_{\text{independent of }p(2)}\\&\qquad+\underbrace{\mathbb{P}\big[\mu^1(c_{-i})=2\big]\big[V\big(p(2)\big)-V(p_1)\big]}_{\le 0}.\end{aligned}$$

Replacing $p(2)$ with $p_1$, the last summand above is equal to zero:

$$\begin{aligned}&V_n\big(p_1,p(2),\dots,p(n)\big)\\&\le V_n\big(p_1,p_1,p(3),\dots,p(n)\big)\\&=\underbrace{\sum_{m=3}^{n}\mathbb{P}\big[\mu^1(c_{-i})=m\big]V\big(p(m)\big)+\left(1-\sum_{m=3}^{n}\mathbb{P}\big[\mu^1(c_{-i})=m\big]\right)}_{\text{independent of }p_1}V(p_1),\end{aligned}$$

which is analogous to (A.23). Now repeat the argument, that is, first replace $p_1$ with $p_2 \in \{p_1, p(3)\}$ such that $V(p_2) = \max\{V(p_1), V(p(3))\}$, then replace $p(3)$ with $p_2$. These two steps yield that

$$V_n\big(p_1, p_1, p(3), \dots, p(n)\big) \leq V_n\big(p_2, p_2, p_2, p(4), \dots, p(n)\big).$$

Continuing this way, we eventually obtain that $V_n\big(p(1), \dots, p(n)\big) \leq V_n(p_{n-1}, \dots, p_{n-1}) = V(p_{n-1})$. Hence, a fixed-price mechanism with price $p_{n-1}$ weakly increases the principal's payoff.

## References


Atlamaz, M., & Yengin, D. (2008). Fair Groves mechanisms. *Social Choice and Welfare*, *31*(4), 573–587.

Ausubel, L. M., Cramton, P., Pycia, M., Rostek, M., & Weretka, M. (2014). Demand reduction and inefficiency in multi-unit auctions. *Review of Economic Studies*, *81*(4), 1366–1400.

Barberà, S., Berga, D., & Moreno, B. (2010). Individual versus group strategy-proofness: When do they coincide? *Journal of Economic Theory*, *145*(5), 1648–1674.

Barberà, S., Berga, D., & Moreno, B. (2016). Group strategy-proofness in private good economies. *American Economic Review*, *106*(4), 1073–1099.

Barberà, S., & Jackson, M. O. (1995). Strategy-proof exchange. *Econometrica*, *63*(1), 51–87.

Bennett, E., & Conn, D. (1977). The group incentive properties of mechanisms for the provision of public goods. *Public Choice*, *29*(2), 95–102.

Bierbrauer, F. J., & Hellwig, M. F. (2016). Robustly coalition-proof incentive mechanisms for public good provision are voting mechanisms and vice versa. *Review of Economic Studies*, *83*(4), 1440–1464.

Bosio, E., & Djankov, S. (2020, February 5). *How large is public procurement?* World Bank Let's Talk Development Blog.

Brenner, M., Galai, D., & Sade, O. (2009). Sovereign debt auctions: Uniform or discriminatory? *Journal of Monetary Economics*, *56*(2), 267–274.

Bu, N. (2016). Joint misrepresentation with bribes. *Economic Theory*, *61*(1), 115–125.

Burkett, J., & Woodward, K. (2020). Uniform price auctions with a last accepted bid pricing rule. *Journal of Economic Theory*, *185*, 104954.

Che, Y.-K., & Kim, J. (2006). Robustly collusion-proof implementation. *Econometrica*, *74*(4), 1063–1107.

Che, Y.-K., & Kim, J. (2009). Optimal collusion-proof auctions. *Journal of Economic Theory*, *144*(2), 565–603.

Chen, J., & Micali, S. (2012). Collusive dominant-strategy truthfulness. *Journal of Economic Theory*, *147*(3), 1300–1312.
Chun, Y., Mitra, M., & Mutuswami, S. (2014). Egalitarian equivalence and strategyproofness in the queueing problem. *Economic Theory*, *56*(2), 425–442.
Deckelbaum, A., & Micali, S. (2017). Collusion, efficiency, and dominant strategies. *Games and Economic Behavior*, *103*, 83–93.
Drexl, M., & Kleiner, A. (2018). Why voting? A welfare analysis. *American Economic Journal: Microeconomics*, *10*(3), 253–271.
Goldberg, A. V., & Hartline, J. D. (2005). Collusion-resistant mechanisms for single-parameter agents. In *SODA '05: Proceedings of the Sixteenth Annual ACM-SIAM Symposium on Discrete Algorithms* (pp. 620–629). Society for Industrial and Applied Mathematics.
Gorokh, A., Banerjee, S., & Iyer, K. (2019, May 7). *When bribes are harmless: The power and limits of collusion-resilient mechanism design*. Unpublished.
Green, J., & Laffont, J.-J. (1979). On coalition incentive compatibility. *Review of Economic Studies*, *46*(2), 243–254.
Grossman, Z., Pincus, J., Shapiro, P., & Yengin, D. (2019). Second-best mechanisms for land assembly and hold-out problems. *Journal of Public Economics*, *175*, 1–16.
Hagen, M., & Hernando-Veciana, Á. (2021). Multidimensional bargaining and posted prices. *Journal of Economic Theory*, *196*, 105317.
Hagerty, K. M., & Rogerson, W. P. (1987). Robust trading mechanisms. *Journal of Economic Theory*, *42*(1), 94–107.
Hashimoto, K., & Saitoh, H. (2015). Strategy-proof cost sharing under increasing returns: Improvement of the supremal welfare loss. *Games and Economic Behavior*, *89*, 101–121.
Holmström, B. (1979). Groves' schemes on restricted domains. *Econometrica*, *47*(5), 1137–1144.
Immorlica, N., Mahdian, M., & Mirrokni, V. S. (2008). Limitations of cross-monotonic cost-sharing schemes. *ACM Transactions on Algorithms*, *4*(2), 1–25.
Juarez, R. (2013). Group strategyproof cost sharing: The role of indifferences. *Games and Economic Behavior*, *82*, 218–239.
Juarez, R. (2018, May 7). *Optimal group strategy-proof cost sharing*. Unpublished.
Kuzmics, C., & Steg, J.-H. (2017). On public good provision mechanisms with dominant strategies and balanced budget. *Journal of Economic Theory*, *170*, 56–69.
Laffont, J.-J., & Martimort, D. (1997). Collusion under asymmetric information. *Econometrica*, *65*(4), 875–911.
Laffont, J.-J., & Martimort, D. (2000). Mechanism design with collusion and correlation. *Econometrica*, *68*(2), 309–342.

Le Breton, M., & Zaporozhets, V. (2009). On the equivalence of coalitional and individual strategy-proofness properties. *Social Choice and Welfare*, *33*(2), 287–309.
Marshall, R. C., & Marx, L. M. (2012). *The economics of collusion*. MIT Press.
Massó, J., & Neme, A. (2007). Bribe-proof rules in the division problem. *Games and Economic Behavior*, *61*(2), 331–343.
McAfee, R. P., & McMillan, J. (1992). Bidding rings. *American Economic Review*, *82*(3), 579–599.
Mehta, A., Roughgarden, T., & Sundararajan, M. (2009). Beyond Moulin mechanisms. *Games and Economic Behavior*, *67*(1), 125–155.
Mihalák, M., Penna, P., & Widmayer, P. (2016). Bribeproof mechanisms for two-values domains. In M. Gairing & R. Savani (Eds.), *Algorithmic Game Theory: 9th International Symposium, SAGT 2016, Liverpool, UK, September 19–21, 2016, Proceedings* (pp. 289–301). Springer.
Milgrom, P. (2004). *Putting auction theory to work*. Cambridge University Press.
Mitra, M., & Mutuswami, S. (2011). Group strategyproofness in queueing models. *Games and Economic Behavior*, *72*(1), 242–254.
Miyagawa, E. (2001). House allocation with transfers. *Journal of Economic Theory*, *100*(2), 329–355.
Mizukami, H. (2003). On the constancy of bribe-proof solutions. *Economic Theory*, *22*(1), 211–217.
Moulin, H. (1999). Incremental cost sharing: Characterization by coalition strategy-proofness. *Social Choice and Welfare*, *16*(2), 279–320.
Moulin, H., & Shenker, S. (2001). Strategyproof sharing of submodular costs: Budget balance versus efficiency. *Economic Theory*, *18*(3), 511–533.
Mukherjee, C. (2013). Weak group strategy-proof and queue-efficient mechanisms for the queueing problem with multiple machines. *International Journal of Game Theory*, *42*(1), 131–163.
Mukherjee, C. (2014). Fair and group strategy-proof good allocation with money. *Social Choice and Welfare*, *42*(2), 289–311.
Mukherjee, C. (2020). On group strategyproof and optimal object allocation. *Economic Theory Bulletin*, *8*(2), 289–304.
Mutuswami, S. (2005). Strategyproofness, non-bossiness and group strategyproofness in a cost sharing model. *Economics Letters*, *89*(1), 83–88.
Pápai, S. (2000). Strategyproof assignment by hierarchical exchange. *Econometrica*, *68*(6), 1403–1433.
Porter, R., Shoham, Y., & Tennenholtz, M. (2004). Fair imposition. *Journal of Economic Theory*, *118*(2), 209–228.
Pountourakis, E., & Vidali, A. (2012). A complete characterization of group-strategyproof mechanisms of cost-sharing. *Algorithmica*, *63*(4), 831–860.
Rochet, J. (1985). The taxation principle and multi-time Hamilton-Jacobi equations. *Journal of Mathematical Economics*, *14*(2), 113–128.

Roughgarden, T., & Sundararajan, M. (2009). Quantifying inefficiency in cost-sharing mechanisms. *Journal of the ACM*, *56*(4), 23.

Satterthwaite, M. A., & Sonnenschein, H. (1981). Strategy-proof allocation mechanisms at differentiable points. *Review of Economic Studies*, *48*(4), 587–597.

Schummer, J. (2000a). Eliciting preferences to assign positions and compensation. *Games and Economic Behavior*, *30*(2), 293–318.

Schummer, J. (2000b). Manipulation through bribes. *Journal of Economic Theory*, *91*(2), 180–198.

Shinozaki, H. (2022, February 25). *No price envy in the multi-unit object allocation problem with non-quasi-linear preferences*. Discussion Paper No. 1164, Institute of Social and Economic Research, Osaka University, Japan.

Thomson, W. (2016). Non-bossiness. *Social Choice and Welfare*, *47*(3), 665–696.

Yengin, D. (2012). Egalitarian-equivalent Groves mechanisms in the allocation of heterogenous objects. *Social Choice and Welfare*, *38*(1), 137–160.

Yengin, D. (2013). Identical preferences lower bound for allocation of heterogenous tasks and NIMBY problems. *Journal of Public Economic Theory*, *15*(4), 580–601.

Yengin, D. (2017). No-envy and egalitarian-equivalence under multi-object-demand for heterogeneous objects. *Social Choice and Welfare*, *48*(1), 81–108.

Young, H. (1994). Cost allocation. In R. Aumann & S. Hart (Eds.), *Handbook of game theory with economic applications* (pp. 1193–1235, Vol. 2). Elsevier.