Collusion-proof and Fair Auctions

Martin Hagen*

Universidad Carlos III de Madrid

September 21, 2019

Abstract

In the standard auction model, we provide a necessary and sufficient condition on the value domain under which non-trivial mechanisms exist that satisfy group strategy-proofness and symmetry. In particular, this condition is satisfied (violated) if values are drawn from a finite set (an interval).

Keywords: mechanism design; social choice theory; auctions; group strategy-proofness; symmetry.

JEL Classification: C72, D44, D71, D82.

*Universidad Carlos III de Madrid, Department of Economics, Calle Madrid 126, 28903 Getafe, Spain. Email: mhagen@eco.uc3m.es. I am thankful for helpful comments from Ángel Hernando Veciana and an anonymous referee. Financial support from the Spanish Ministry of Economy and Competitiveness, grant ECO2015-68406-P, is gratefully acknowledged.

1 Introduction

Collusion-proofness and fairness are desirable, but possibly conflicting, properties in mechanism design. In this note, we study whether they are compatible in the standard auction model, where a single good is up for sale to agents who have private information on how much they value the good. Following the literature on strategy-proofness, we formalize collusion-proofness and fairness through *group strategy-proofness* and *symmetry*, respectively. Group strategy-proofness requires that no group of agents have a deviation from their dominant strategies that makes at least one group member better off and no other group member worse off. Symmetry requires that any two agents with the same value obtain the same payoff.

Group strategy-proofness and symmetry are clearly satisfied by a mechanism that never assigns the good and charges no payments—but such *trivial* mechanisms are hardly appealing. We consider a mechanism to be *non-trivial* if at least one agent's utility is not constant. Non-trivial mechanisms that satisfy either group strategy-proofness or symmetry do exist. For example, the second-price auction is symmetric but not group strategy-proof, whereas a posted-price mechanism in which the auctioneer offers the good to one of the agents at an exogenous price is group strategy-proof but not symmetric. It is far from obvious, however, whether non-trivial mechanisms can satisfy both properties simultaneously.

Our contribution is to show that non-trivial, group strategy-proof and symmetric mechanisms exist if and only if the value domain has a maximum and this maximum is an isolated point. In particular, this condition holds for any finite domain but is violated under the standard assumption that values are drawn from an interval. While it is often a matter of convenience whether values are taken to be discrete or continuous, our result illustrates that this modeling choice can lead to different conclusions.

2 Related Literature

Within the vast literature on strategy-proofness (Barberà, 2011), three contributions are particularly related to our analysis because they provide similar impossibility results in a model with a single good and quasi-linear utility. First, Mukherjee (2014) shows that group strategy-proofness and anonymity (in utility terms) are incompatible. Second, Kato, Ohseto, and Tamura (2015) show that strategy-proofness, symmetry and budget balance are incompatible. Third, Fujinaka and Sakai (2007) show that Maskin monotonicity and weak symmetry are incompatible. Common to all three papers is the assumption that the auctioneer cannot withhold the good; and this feature is critically exploited in the proofs. Our model, by contrast, allows the good to be unassigned, which not only affects the technicalities behind the impossibility result but also opens the door to a new possibility result.

The option to withhold the good is also important for axiomatizations of Vickrey auctions with a reserve price (Sakai, 2013; Mukherjee and Basu, 2018). These mechanisms do not satisfy group strategy-proofness, however, and thus fall outside the realm of our analysis.

Related to our possibility result is Ando, Kato, and Ohseto's (2008) finding that strategy-proof, weakly symmetric and budget-balanced mechanisms exist on a particular domain with at most as many values as there are agents in the economy. Few other papers have shown that impossibility results can sometimes be overcome by discretizing the preference domain. Most notably, Myerson and Satterthwaite's (1983) famous impossibility result does not generally hold with finitely many values (Matsuo, 1989; Kos and Manea, 2009; Othman and Sandholm, 2009; Flesch, Schröder, and Vermeulen, 2016).

3 Model

An auctioneer (female) wants to sell one indivisible good to one of $n \geq 2$ agents (male). The set of agents is $N \equiv \{1, 2, \dots, n\}$. The auctioneer, who is labeled with 0, has the option to keep the good. Each agent $i \in N$ has a **value** $v_i \in \mathbb{R}$ for the good. The **value domain** $V \subseteq \mathbb{R}$ contains at least two elements. The supremum of V is denoted by $\bar{v} \equiv \sup\{V\}$. Note that values can be negative, so the model also accommodates procurement auctions. Given $v_i \in V$ for all $i \in N$, let $v \equiv (v_i)_{i \in N}$ be the **value profile** of all agents. For any group of agents $G \subseteq N$, define $v_G \equiv (v_i)_{i \in G}$, $v_{-G} \equiv (v_i)_{i \in N \setminus G}$ and $(v_G, v_{-G}) \equiv v$. Throughout the paper, we eschew set notation for singletons by writing v_{-i} instead of $v_{-\{i\}}$, $G \setminus i$ instead of $G \setminus \{i\}$, and the like. Utility is assumed to be quasi-linear in money; so if agent $i \in N$ with value $v_i \in V$ gets quantity $q_i \in \{0, 1\}$ and transfer $t_i \in \mathbb{R}$, his utility is $v_i q_i + t_i$.

4 Definitions

For all value profiles $v \in V^n$, a **mechanism** specifies the **winner** $w(v) \in N \cup 0$ of the good as well as the monetary **transfer** $t_i(v) \in \mathbb{R}$ that agent $i \in N$ receives.[1] The auctioneer keeps the good whenever $w(v) = 0$. Let $q_i(v) \in \{0, 1\}$ denote the **quantity** of agent $i \in N$ at value profile $v \in V^n$; that is, $q_i(v) \equiv \mathbb{1}\{w(v) = i\}$. For any given mechanism, $u_i(\hat{v}|v_i) \equiv v_i q_i(\hat{v}) + t_i(\hat{v})$ represents the **utility** that agent $i \in N$ with value $v_i \in V$ obtains when the reported value profile is $\hat{v} \in V^n$. If agent $i \in N$ is truthful (i.e. $\hat{v}_i = v_i$), we write $u_i^*(\hat{v}) \equiv u_i(\hat{v}|v_i)$.

A mechanism is **strategy-proof** (**SP**) if reporting truthfully is always a best response for each agent; that is, $u_i^*(v) \geq u_i(\hat{v}_i, v_{-i}|v_i)$ for all $i \in N$, $v \in V^n$ and $\hat{v}_i \in V$. A mechanism is **group strategy-proof** (**GSP**) if every joint misreport by a group of agents that makes a group member better off also makes another group member worse off; that is, for all $G \subseteq N$, $v \in V^n$ and $\hat{v}_G \in V^{|G|}$ such that $u_i^*(v) < u_i(\hat{v}_G, v_{-G}|v_i)$ for some $i \in G$, it holds that $u_j^*(v) > u_j(\hat{v}_G, v_{-G}|v_j)$ for some $j \in G$. Note that group strategy-proofness implies strategy-proofness. A mechanism is **symmetric** (**SYM**) if any two agents with the same value get the same (truthful) utility; that is, $u_i^*(v) = u_j^*(v)$ for all $v \in V^n$ and $i, j \in N$ such that $v_i = v_j$. A mechanism is **trivial** if all agents receive the same utility at all value profiles; that is, there exists $u \in \mathbb{R}$ such that $u_i^*(v) = u$ for all $i \in N$ and $v \in V^n$. Note that the utilities of a trivial mechanism can be reproduced by never assigning the good and paying u to each agent. Conversely, a mechanism is **non-trivial** if $u_i^*(v) \neq u_i^*(\hat{v})$ for some $i \in N$ and $v, \hat{v} \in V^n$.

5 Result

A necessary and sufficient condition for the existence of non-trivial, group strategy-proof and symmetric mechanisms is that $\bar{v}$ is an isolated point of V (under the standard topology on $\mathbb{R}$).

Theorem. *Non-trivial mechanisms that satisfy group strategy-proofness and symmetry exist if and only if* $\sup\{V \setminus \bar{v}\} < \bar{v}$.

[1] We are restricting attention to mechanisms that are *direct* and *deterministic*. By the revelation principle (Gibbard, 1973), this is without loss of generality if we are interested in deterministic mechanisms that have a pure dominant-strategy equilibrium.

As the proof below will reveal, the Theorem also holds under stronger fairness conditions, such as anonymity (in utility terms) and envy-freeness. The result is stated with respect to symmetry because this is the weakest property under which we are able to establish the impossibility result ("only if"), which is the more challenging of the two directions.

6 Proof of Sufficiency

Assume that $\sup\{V \setminus \bar{v}\} < \bar{v}$, which implies that $\bar{v} \in V$. We now construct a non-trivial mechanism that satisfies group strategy-proofness and symmetry: If no agent reports value $\bar{v}$, the auctioneer keeps the good and executes no payments. Otherwise, she assigns the good arbitrarily to one of the agents who report $\bar{v}$, requests the winner to pay p such that $\sup\{V \setminus \bar{v}\} < p < \bar{v}$, and transfers $\bar{v} - p$ to each loser. Hence, the truthful utility of agent $i \in N$ at value profile $v \in V^n$ is given by

$$u_i^*(v) = \begin{cases} 0 & \text{if } v_j < \bar{v} \text{ for all } j \in N \\ \bar{v} - p & \text{otherwise.} \end{cases}$$

This mechanism is **symmetric** because, at each value profile, all agents receive the same utility. In fact, stronger fairness conditions such as anonymity (in utility terms) and envy-freeness are satisfied. The mechanism is also **non-trivial** because $\bar{v} - p > 0$. Finally, it is **group strategy-proof**: If the good is assigned, all agents receive the largest utility attainable $(\bar{v} - p)$. Incentives to lie—individually or collectively—can thus only arise if the good is withheld at the true value profile $v \in V^n$, which occurs if and only if $v_i < \bar{v}$ for all $i \in N$. To change the outcome, a (joint) misreport $\hat{v} \in V^n$ must involve an agent $i \in N$ who reports $\hat{v}_i = \bar{v}$ and wins. This deviation, however, makes i worse off because $u_i(\hat{v}|v_i) = v_i - p \leq \sup\{V \setminus \bar{v}\} - p < 0 = u_i^*(v)$. Hence, there are no profitable deviations.

7 Proof of Necessity

In this section, we show that $\sup\{V \setminus \bar{v}\} < \bar{v}$ is necessary for the existence of non-trivial mechanisms that satisfy group strategy-proofness and symmetry.

7.1 Preliminary Results

We start with two preliminary results. The first one states the well-known monotonicity and continuity properties of strategy-proof mechanisms. For brevity, we write "Lemma 1 (SP)" to indicate that Lemma 1 holds under strategy-proofness.

Lemma 1 (SP)**.** *For all $i \in N$ and $v_{-i} \in V^{n-1}$,*

(i) $q_i(\cdot, v_{-i})$ is weakly increasing;

(ii) $u_i^*(\cdot, v_{-i})$ is continuous.

Proof. See, for instance, Börgers (2015, Lemmas 2.1 and 2.2). □

Under group strategy-proofness, the monotonicity of Lemma 1(i) partially extends from individuals to groups: if the auctioneer keeps the good at value profile v, then she also keeps it at any other value profile $\hat{v}$ where all agents appreciate the good less. This is a weak version of *Maskin monotonicity* (Maskin, 1999) and captured by the first part of the following lemma.[2]

Lemma 2 (GSP)**.** *For all $v, \hat{v} \in V^n$,*

(i) if $w(v) = 0$ and $v_i > \hat{v}_i$ for all $i \in N$, then $w(\hat{v}) = 0$;

(ii) if $w(v) = w(\hat{v})$, then $t_i(v) = t_i(\hat{v})$ for all $i \in N$.

Proof. See, for instance, Schummer (2000, Theorem 3). □

The second part of Lemma 2 says that the agents' transfers can only vary across value profiles if the winner of the good changes. This result encapsulates the extent of price rigidity necessary to prevent coalitional deviations. An important implication of Lemma 2(ii) is that, for all $i \in N$, there exists a function $\tau_i \colon N \cup 0 \to \mathbb{R}$ such that $t_i(v) = \tau_i(w(v))$ for all $v \in V^n$. Thus, $\tau_i(j)$ denotes the payment that agent $i \in N$ receives at any value profile where $j \in N \cup 0$ wins the good. For convenience, define $\tau(0) \equiv \tau_1(0)$.

[2]Some authors (e.g. Fujinaka and Sakai, 2007) use a stronger definition of Maskin monotonicity which allows the agents' values to remain unchanged (that is, the inequalities in Lemma 2(i) are weak). In our model, this stronger notion is *not* implied by group strategy-proofness.

7.2 Core of the Proof

Based on the previous results, we now show that every group strategy-proof and symmetric mechanism is trivial if $\sup\{V \setminus \bar{v}\} = \bar{v}$. This equation says that V either has no upper bound or has an accumulation point at $\bar{v}$. The first step is to prove that no agent wins the good at more than one value profile where all agents have the same value.

Lemma 3 (GSP+SYM)**.** *For each $i \in N$, there is at most one $v \in V^n$ such that $w(v) = i$ and $v_j = v_i$ for all $j \in N \setminus i$.*

Proof. Consider any $i \in N$ and $v, \hat{v} \in V^n$ such that $w(v) = w(\hat{v}) = i$ as well as $v_j = v_i$ and $\hat{v}_j = \hat{v}_i$ for all $j \in N \setminus i$. Symmetry at v and $\hat{v}$ requires that $v_i + \tau_i(i) = \tau_j(i)$ and $\hat{v}_i + \tau_i(i) = \tau_j(i)$ for all $j \in N \setminus i$. It follows that $v_i = \hat{v}_i$ and thus $v = \hat{v}$. □

Next, we show that the auctioneer keeps the good at all interior value profiles, where no agent has the maximum value.

Lemma 4 (GSP+SYM)**.** *Suppose* $\sup\{V \setminus \bar{v}\} = \bar{v}$*. For all $v \in V^n$ such that $v_i < \bar{v}$ for all $i \in N$, $w(v) = 0$ and $t_i(v) = \tau(0)$ for all $i \in N$.*

Proof. Consider any $v \in V^n$ such that $v_i < \bar{v}$ for all $i \in N$. Since $\sup\{V \setminus \bar{v}\} = \bar{v}$, there are infinitely many $\hat{v} \in V^n$ such that $\max\{v_i : i \in N\} < \hat{v}_1 = \cdots = \hat{v}_n \leq \bar{v}$. By Lemma 3, $w(\hat{v}) = 0$ for almost all of these $\hat{v}$. Hence, by Lemma 2(i), $w(v) = 0$. Moreover, Lemma 2(ii) and symmetry imply that $t_i(v) = t_i(\hat{v}) = t_1(\hat{v}) = \tau(0)$ for all $i \in N$. □

If V is right-open (i.e. $\bar{v} \notin V$), all value profiles are interior. Hence, by Lemma 4, $u_i^*(v) = \tau(0)$ for all $i \in N$ and $v \in V^n$, so the mechanism is trivial. If V is right-closed (i.e. $\bar{v} \in V$), the proof becomes more involved because the good can now be assigned at boundary value profiles, where $v_i = \bar{v}$ for some $i \in N$. Building on the continuity inherent to strategy-proofness (Lemma 1(ii)), our final two lemmas show that the agents' utilities at the boundaries are as if the good were not assigned—which means triviality.

Lemma 5 (GSP+SYM)**.** *Suppose* $\sup\{V \setminus \bar{v}\} = \bar{v}$*. For all $v \in V^n$ and $i \in N$ such that $v_i = \bar{v}$, $u_i^*(v) = \tau(0)$.*

Proof. Let $G \equiv \{i \in N : v_i = \bar{v}\} \neq \emptyset$ be the set of agents who have the maximum value at $v \in V^n$. We show that $u_i^*(v) = \tau(0)$ for all $i \in G$. The argument depends on whether $w(v) \in G$ or $w(v) \notin G$; but since both cases are similar, we only present the former. Accordingly, let $j \equiv w(v) \in G$. By symmetry, $\bar{v} + \tau_j(j) = \tau_k(j)$ for all $k \in G \setminus j$. If $\tau(0) < \bar{v} + \tau_j(j)$, then $\sup\{V \setminus \bar{v}\} = \bar{v}$ guarantees the existence of $\hat{v}_j \in V$ such that $\tau(0) - \tau_j(j) < \hat{v}_j < \bar{v}$. Letting $\hat{v}_k < \bar{v}$ for all $k \in G \setminus j$, Lemma 4 implies that $w(\hat{v}_G, v_{-G}) = 0$. Hence, $u^*_j(\hat{v}_G, v_{-G}) = \tau(0) < \hat{v}_j + \tau_j(j) = u_j(v|\hat{v}_j)$ and $u^*_k(\hat{v}_G, v_{-G}) = \tau(0) < \bar{v} + \tau_j(j) = \tau_k(j) = u_k(v|\hat{v}_k)$ for all $k \in G \setminus j$, which violates group strategy-proofness at $(\hat{v}_G, v_{-G})$. Similarly, $\tau(0) > \bar{v} + \tau_j(j)$ conflicts with group strategy-proofness at v. Therefore, $\tau(0) = \bar{v} + \tau_j(j) = \tau_k(j)$ for all $k \in G \setminus j$, resulting in $u^*_i(v) = \tau(0)$ for all $i \in G$. □

Lemma 6 (GSP+SYM)**.** *Suppose* $\sup\{V \setminus \bar{v}\} = \bar{v}$*. For all* $v \in V^n$ *and* $i \in N$*,* $u^*_i(v) = \tau(0)$*.*

Proof. If $\bar{v} \notin V$, the claim follows immediately from Lemma 4. Assume, then, that $\bar{v} \in V$. Considering any $i \in N$ and $v_{-i} \in V^{n-1}$, we prove that $u^*_i(\cdot, v_{-i}) = \tau(0)$. Define $v \equiv (\bar{v}, v_{-i})$. Since $v_i = \bar{v}$, Lemma 5 states that $u^*_i(v) = \tau(0)$. If $i \neq w(v)$, strategy-proofness requires that $q_i(\cdot, v_{-i}) = 0$ and thus $u^*_i(\cdot, v_{-i}) = u^*_i(v) = \tau(0)$, as desired. Finally, if $i = w(v)$, a slight variation of the argument behind Lemma 5 shows that $q_i(\hat{v}_i, v_{-i}) = 0$ for all $\hat{v}_i \in V$ such that $\hat{v}_i < \bar{v}$. Hence, by strategy-proofness, $u^*_i(\hat{v}_i, v_{-i}) = \text{const.}$ for all $\hat{v}_i \in V$ such that $\hat{v}_i < \bar{v}$. Since $\sup\{V \setminus \bar{v}\} = \bar{v}$, the continuity of $u^*_i(\cdot, v_{-i})$ at $v_i = \bar{v}$ implies that $u^*_i(\cdot, v_{-i}) = u^*_i(v) = \tau(0)$. □

8 Conclusion

In the standard auction model, we have provided a necessary and sufficient condition for the existence of non-trivial mechanisms that satisfy group strategy-proofness and symmetry. Naturally, the next step is to fully characterize these mechanisms when they exist—in order to assess how valuable the possibility result actually is.

References

Ando, K., Kato, M., & Ohseto, S. (2008). Strategy-proof and symmetric allocation of an indivisible good. *Mathematical Social Sciences*, *55*(1), 14–23.

Barberà, S. (2011). Strategyproof social choice. In K. J. Arrow, A. Sen, & K. Suzumura (Eds.), *Handbook of social choice and welfare* (Vol. 2, pp. 731–831). Amsterdam, The Netherlands: North-Holland (Elsevier).

Börgers, T. (2015). *An introduction to the theory of mechanism design*. New York, NY: Oxford University Press.

Flesch, J., Schröder, M., & Vermeulen, D. (2016). Implementable and ex-post IR rules in bilateral trading with discrete values. *Mathematical Social Sciences*, *84*, 68–75.

Fujinaka, Y., & Sakai, T. (2007). Maskin monotonicity in economies with indivisible goods and money. *Economics Letters*, *94*(2), 253–258.

Gibbard, A. (1973). Manipulation of voting schemes: A general result. *Econometrica*, *41*(4), 587–601.

Kato, M., Ohseto, S., & Tamura, S. (2015). Strategy-proofness versus symmetry in economies with an indivisible good and money. *International Journal of Game Theory*, *44*(1), 195–207.

Kos, N., & Manea, M. (2009, September). *Efficient trade mechanism with discrete values*. Unpublished Manuscript.

Maskin, E. (1999). Nash equilibrium and welfare optimality. *Review of Economic Studies*, *66*(1), 23–38.

Matsuo, T. (1989). On incentive compatible, individually rational, and ex post efficient mechanisms for bilateral trading. *Journal of Economic Theory*, *49*(1), 189–194.

Mukherjee, C. (2014). Fair and group strategy-proof good allocation with money. *Social Choice and Welfare*, *42*(2), 289–311.

Mukherjee, C., & Basu, R. (2018). *Characterization of Vickrey auction with reserve price for multiple objects* (Indian Institute of Management Calcutta Working Paper No. 816).

Myerson, R. B., & Satterthwaite, M. A. (1983). Efficient mechanisms for bilateral trading. *Journal of Economic Theory*, *29*(2), 265–281.

Othman, A., & Sandholm, T. (2009). How pervasive is the Myerson-Satterthwaite impossibility? In C. Boutilier (Ed.), *Proceedings of the 21st International Joint Conference on Artifical Intelligence (IJCAI-09)* (pp. 233–238). Menlo Park, CA: AAAI Press.

Sakai, T. (2013). Axiomatizations of second price auctions with a reserve price. *International Journal of Economic Theory*, *9*(3), 255–265.

Schummer, J. (2000). Eliciting preferences to assign positions and compensation. *Games and Economic Behavior*, *30*(2), 293–318.