Repeated Contests with Commitment Types*

Stefano Barbieri†, Marco Serena‡

May 5, 2025

Abstract

Contestants fight in repeated contests and privately know whether they are rational utility maximizers or automatons committed to always fighting "all-in." In the unique symmetric equilibrium, rational contestants fight all-in in early contests as doing so buys a beneficial "all-in look" that intimidates rivals in future contests. In equilibrium, several structural differences emerge between periods in which multiple contestants maintain an all-in look, as opposed to one: most prominently, *only* in the former case aggregate efforts may exceed the per-period prize, and *only* in the latter payoffs can be positive.
JEL classification codes: C72, D44, D82, D83.

*We thank the Associate Editor and two anonymous referees of this journal for their constructive criticism. We are also pleased to acknowledge useful comments byJordan Adamson, Benjamin Blumenthal, Subhashish Chowdhury, Mikhail Drugov, Amanda Friedenberg, Alice Hallman, Kai Konrad, Dan Kovenock, Marc Möller, and Marta Troya-Martinez. The authors would like to extend their sincere appreciation to the Max Planck Institute for Tax Law and Public Finance for the hospitality extended during spring of 2024, when part of this work was carried out. We would like to thank participants at the SAET 2021, GAMES 2021, Max Planck Summer School on the Political Economy of Conflict and Redistribution 2021, Contest: Theory and Evidence conference 2021, the Bavarian Micro Day 2021, the Winter Meeting of the Econometric Society 2021 and 2022, PET 2022, Conference of Economic Design 2022, RES Sympium 2022, XXX EWET 2022, CMiD 2022, 33^{rd} Stony Brook conference on Game Theory, SEAT 2022, EARIE 2022, Jornadas de Economia Industrial 2022, VfS 2022, Global seminar on Contests & Conflicts 2022, ASSET 2022, ASSA 2022, and at many seminars. Marco Serena gratefully acknowledges financial support provided by Comunidad de Madrid, Spain, under grant 2024-T1/PH-HUM-31333. All errors are our own.
†Tulane University, New Orleans. *Email*: sbarbier@tulane.edu
‡CUNEF Universidad, Madrid. *Email*: marco.serena@cunef.edu

1 Introduction

In repeated contests, a contestant's high effort today may not only increase her probability of winning today's battle, but also help build a deterring reputation for future battles. For instance, organized criminal groups fighting over the control of illegal markets—such as the multi-billion dollar industry of smuggling drugs, firearms, liquor, or people—often employ acts of particularly heinous violence not only to resolve today's dispute, but also to buy a tough look in the eyes of their rivals, a strategy that may be beneficial in *future* disputes (e.g., MacCoun and Reuter, 2001; Livingston, 2011; Shackelford and Weekes-Shackelford, 2012). Similarly, in litigation, parties choosing particularly high legal expenditures in pursuing today's case build a litigious look that is typically beneficial in future litigations (Allison et al., 2010; Hovenkamp, 2013).[1] We provide a simple theoretical framework to analyze how a tough look affects repeated contests in which a contestant, by fighting particularly hard today, buys a tough look for the future.

We focus on a simple setup that captures the incentives to buy a tough look in repeated contests: n contestants fight repeatedly in a sequence of T contests, one in each of T periods. Each contest is modeled as a standard all-pay auction with a fixed prize of value 1; that is, contestants simultaneously exert efforts and the contestant who exerts the highest effort wins 1. The cost of effort equals the effort level itself, and it is paid regardless of victory or defeat. Each contestant is privately informed about her persistent type, which is either a standard utility-maximizing rational type or, with probability ε, an "all-in" automaton locked into exerting "all-in effort" equal to 1 in all periods. As the per-period prize is 1, setting automatons' efforts to 1 is a simple way to capture the incentives to buy a tough look by exerting the all-in effort (e.g., particularly heinous violence in a turf war). Efforts are observable. Thus, exerting a "non-all-in" effort (i.e., smaller than 1) immediately and forever unmasks a contestant as rational. A contestant who fights all-in in today's contest builds an all-in look in the eyes of her rivals who are uncertain whether she is an all-in automaton, or a rational type trying to buy an all-in look. We hence borrow the standard approach to reputation through commitment (Kreps and Wilson, 1982; Milgrom and Roberts, 1982; and the extensive literature analysis of Mailath and Samuelson, 2006); our contribution is its first (to the best of our knowledge) application to repeated contests.

We fully characterize the unique type-symmetric equilibrium of the (T, n, ε)-game. In particular, we find that several structural differences emerge between periods in which: 1) multiple contestants have an all-in look, as they all have always fought all-in thus far

[1] Tingley and Walter (2011) provides experimental evidence that subjects do invest in building reputation for toughness in repeated games.

(henceforth, an "all-in oligopoly"), and 2) only one contestant has an all-in look, as only she has always fought all-in thus far (henceforth, an "all-in monopoly"). When confusion does not arise, in what follows, we focus on rational contestants and omit the qualifier "rational". Furthermore, we refer to the cumulative payoff in a period as the sum of present and all future per-period payoffs.

In a period of all-in monopoly, the monopolist is not afraid of being up against all-in automatons. Therefore, in equilibrium, the monopolist enjoys a strictly positive cumulative payoff because others fear that she is an all-in automaton and their efforts are thus discouraged. Importantly, in any period, a contestant has a strictly positive cumulative equilibrium payoff if and only if she is an all-in monopolist; in all other cases, cumulative equilibrium payoffs are zero. The value of such a strictly positive payoff is endogenously determined by balancing the monopolist's incentives to boost her reputation for the future and to cash in on her reputational advantage. Furthermore, as the expected aggregate effort in an all-in monopoly is lower than in the natural benchmark without all-in automatons ($\varepsilon = 0$), one can say that all-in looks, when monopolized, *discourage* efforts.

In a period of all-in oligopoly, in equilibrium, contestants with an all-in look engage in a fierce fight because the stakes include the per-period prize *and* the prospect of becoming an all-in monopolist in the future and obtaining the monopolist's strictly positive cumulative equilibrium payoff. For this reason, the expected aggregate effort may be higher than in the benchmark without all-in automatons; thus, all-in looks may *encourage* efforts, in an all-in oligopoly. Because of the fierce fight among all-in oligopolists, per-period expected equilibrium payoffs of contestants who fight all-in are negative and the expected aggregate effort in a period with an all-in oligopoly may be larger than the per-period prize.

A crucial role is played by equilibrium dynamics. Starting from a period of all-in oligopoly, the equilibrium may follow a path such that contestants fighting all-in incur per-period losses in the hope that their all-in look will eventually outlast that of all their rivals. Along this path, beliefs that others are all-in automatons increase over time until the net cost of fighting all-in becomes too large, because the risk of facing all-in automatons is too high: the fear that others are all-in automatons dominates the prospect that one's all-in look will outlast the rivals'. When such fear dominates, contestants with an all-in look stop fighting all-in and are all unmasked as rational. Thus, the fierce fight in the all-in oligopoly may end suddenly without passing through an all-in monopoly. This dynamic explains why an all-in oligopoly has a maximum endogenous duration; in contrast, the monopolist may fight all-in up until the very last period because she is never afraid of being up against all-in automatons.

Structure of the Paper. Section 2 discusses the related literature. Section 3 describes the model. Section 4 provides illustrative examples that gradually build intuition. Section 5 formally characterizes and analyzes the unique type-symmetric equilibrium with any number of periods and contestants (i.e., the general (T, n, ε)-game), highlighting the structural equilibrium differences between a period in which zero, one, or multiple contestants have an all-in look. Section 6 discusses extensions and robustness of our results. Section 7 concludes.

2 Related Literature

To the best of our knowledge, we are the first to equip a standard repeated contest (all-pay auction) with commitment types. Four features of our simple framework relate to four strands of the literature: repeated contests with private information, war-of-attrition games, repeated games with committed types, and the analysis of aggregate effort in contests.

First, in our setup, actions signal types and contestants update their beliefs about types over time.[2] This feature is shared by models of contests with private information about valuations, abilities, or effort costs rather than about fighting postures (rationality/all-in), which is instead our focus.[3] One-shot two-player contests where, before the contest, a contestant can send a costly signal to her rival are studied by Katsenos (2010), Fu et al. (2013), and Denter et al. (2022) with one-sided and Heijnen and Schoonbeek (2017) with two-sided asymmetric information.[4] Signaling in twice-repeated contests is studied by Catepillán et al. (2022) with one-sided and Münster (2009) and Kubitz (2022) with two-sided asymmetric information over ability (or prize valuation). Signaling in those models is often two-directional: weak types may want to appear strong and strong types may want to appear weak. Two-directional signaling complicates the analysis: none of those models go beyond two periods or two contestants.[5] Our approach with commitment types matches applications where one wants to appear tough, rather than weak. It also gives us enough tractability to fully characterize the equilibrium for any number of periods and contestants. Our T-period n-contestant

[2] For experimental evidence that subjects do understand and react to the fact that players' actions signal privately known types (strength) in contests, see, for example, Konrad and Morath (2018). Beccuti and Möller (2022) consider a common-value best-of-three contest where players observing battle outcomes update their beliefs about the prize value. In a related strand, contestants share verifiable information prior to the contest (e.g., Kovenock et al., 2015; Wu and Zheng, 2017; Ewerhart and Lareida, 2024).

[3] Abreu and Gul (2000; p. 86) provide a well-aimed description of this structural difference between the two families of models: models with canonical private information are "concerned with uncertainty about 'fundamentals'." In contrast, models adopting the committed-type approach are "rather different in that [they] seek to model uncertainty about the strategic intent or strategic posture of the opponent rather than uncertainty about such concrete factors as seller's costs of production or buyer's valuations." We believe the committed-type approach is appropriate to capture the dynamics of building all-in looks in the applications in the Introduction.

[4] See Fu (2006) for a two-player contest where the informed contestant moves earlier than her uninformed rival, and her effort signals her private type.

[5] Krähmer (2007) models t-period two-player repeated contests with binary efforts and learning about contestants' relative abilities. However, players neither have private nor asymmetric information.

results show that going beyond the two-period, two-contestant model is informative. For instance, having more than two contestants gives rise to partial participation and parallel competition between contestants, some with and some without an all-in look. Having more than two periods gives rise to interesting dynamics that cannot be captured in a two-period setup; in fact, 1) the very first period is special because contestants have not yet had the chance of investing into their all-in looks, and 2) the very last period is special because contestants have no incentive to further invest in a costly all-in look for the future.

Second, an equilibrium path of our setup shares common features with war-of-attrition games. In particular, in a period of all-in oligopoly, we find that contestants with an all-in look will keep on fighting all-in with strictly positive probability in the hope of outlasting all rivals (i.e., becoming an all-in monopolist), even if fighting all-in yields a strictly negative current-period payoff. These equilibrium dynamics resemble war-of-attrition games, where contestants typically choose a time to stop and trade off the gains from outlasting other contestants (i.e., stopping later) and the costs incurred as time goes by.[6] In our setup, a war of attrition endogenously arises in equilibrium, due to the interaction between the specific stage game (a standard all-pay auction) and our approach with committed "all-in" types. To the contrary of typical war-of-attrition games, rather than assuming that outlasting the others yields an exogenous benefit, we find that in equilibrium contestants try to outlast all rivals and obtain the (strictly positive) endogenous payoff of an all-in monopolist *because* in all other contingencies payoffs are zero.

Third, our paper is related to the literature on repeated games with committed types.[7] Within this literature, an important strand is that on reputational bargaining. Consider the seminal work of Abreu and Gul (2000, henceforth AG) as an example to illustrate the key differences with our setup. The stage game in AG is a two-player dividing-a-dollar game with sequential endogenous offers, rather than a contest with simultaneous endogenous efforts; the costs arise from discounting in AG, and from efforts themselves in our setup. This highlights the key difference: our stage game (a contest) allows us to analyze the key variable of interest of our paper: the expected efforts. Importantly, while in AG players with reputation essentially choose only the acceptance probability in every period, in our setup contestants

[6] War of attrition games are pioneered by Maynard Smith (1974) and applied to a variety of situations: patent races (Fudenberg et al., 1983), bargaining (Ordover and Rubinstein 1986), public good provision (Bliss and Nalebuff, 1984), and price wars and exit in oligopolistic markets (Fudenberg and Tirole 1986). For a general analysis of wars of attrition, see Bulow and Klemperer (1999).

[7] We are aware of only one paper adopting the committed-type approach to repeated contests—Hovenkamp (2013)—which analyzes a t-period model of repeated litigation; however, his stage game and thus reputation dynamics are structurally different from our setup. In particular, he analyzes litigations by modeling PAEs as long-term players proposing a settlement to short-term players (firms) who can accept or reject. Following a rejection, the long-term player litigates the claim or gives up. Short-term players are either rational or an "impressionable type" that is intimidated whenever the long-term player engages in litigation.

choose not only their probability of mimicking committed types by choosing all-in efforts, but also the continuous *distribution* of non-all-in efforts, which varies in every period and allows us to draw conclusions on expected equilibrium efforts. Hence, our contribution to this literature is not in developing new results on reputation in repeated games, but in being the first to apply reputation building with committed types to repeated contests and analyse the resulting equilibrium efforts, which are the conventional variables of interest in contests (see, e.g., Konrad, 2009; Dechenaux, Kovenock, and Sheremeta, 2015; Fu and Wu, 2019).

Fourth, as mentioned above, we find that per-period aggregate efforts can be larger than 1 in early periods, because contestants with an all-in look fight not only for the per-period prize of 1, but also for becoming the all-in monopolist. This result is reminiscent of the literature on overdissipation (aggregate efforts being larger than the prize) in all-pay auctions, where the typical finding is that overdissipation emerges in sufficiently early periods and appears to decrease over time (see, e.g., Lugovskyy et al., 2010).

The closest paper to ours is Kwiek (2011), who considers a repeated second-price auction with committed types and entry fees. The entry fee gives bidders a signaling motive to bid high in order to deter future bids, which is similar to our reputation motive of "all-in efforts." Nonetheless, our stage game differs from that of Kwiek in that players pay their effort regardless of the battle's outcome: we consider an all-pay, rather than a second-price, auction. Technically, Kwiek's stage game satisfies the condition for being a strictly conflicting interest game (see, Cripps et al., 2005), whereas our stage game does not. Also, in our setup, losing reputation is necessary for obtaining a positive per-period surplus, while in Kwiek's game, the monopolist continues to consume reputation in (almost) all future periods.

3 The Model

In each period $t \in \{1, ..., T\}$ with $2 \leq T < \infty$, a fixed set of $n \geq 2$ contestants simultaneously exert non-negative efforts.[8] In each period, the contestant exerting the highest effort wins a per-period prize of value 1, while the losers obtain 0; ties are broken evenly. Effort costs are identical to efforts and paid by all contestants. Contestants are risk neutral (the per-period payoff equals the prize won, if any, minus effort) and do not discount future payoffs.[9] Recall that, in contrast to the per-period payoff, in what follows, we refer to a contestant's cumulative payoff in period t as the sum of all per-period payoffs from t to T. After each

[8] We assume $T < \infty$ so as to abstract from collusive agreements. In addition, we make the restrictive (for applications) assumption that all contestants know T, so as to focus on reputation over toughness as the only source of uncertainty. Finally, we assume that the set of contestants stays the same across periods and that they are all ex-ante symmetric: these assumptions are restrictive and discussed in Section 6.

[9] We discuss the effects of discounting in Section 6.

period, contestants observe all efforts.

We assume that each contestant is an all-in automaton with ex-ante probability $\varepsilon \in [0, 1)$, and rational with probability $1 - \varepsilon$. Contestants' types are realized once and for all at the beginning of the game and are privately known. A rational contestant is a standard forward-looking payoff maximizer. An all-in automaton is locked into always exerting the all-in effort, equal to 1.[10] As the behavior of all-in automatons is fixed, we focus on the analysis of rational contestants in what follows and omit the qualifier "rational" except when needed to avoid confusion.

In each period t, contestants form beliefs about each other's probability of being all-in automatons. Beliefs depend on the full history of observed (all-in or non-all-in) efforts. In particular, in any Perfect Bayesian Equilibrium,[11] if a contestant always fought all-in until period t, then she is believed by others to be an all-in automaton with a strictly positive probability given by Bayes' rule; if so, we say that she "has an all-in look" in period t. Conversely, if a contestant ever exerted at least once a non-all-in effort (regardless of whether such effort is on- or off-the-equilibrium path), she is unmasked as rational, and we say that she does not have an all-in look in period t (and onwards).[12] We focus on type-symmetric Perfect Bayesian Equilibrium (TSPBE): in any period t, for any pair (i, j), if all players other than i and j believe that i and j are equally likely to be automatons (i and j have "the same all-in look"), then i and j use the same period-t equilibrium strategy. Note that, if multiple contestants have an all-in look in period t, then they have the same all-in look; we call the corresponding belief level $\varepsilon_t > 0$.[13] Hence, when studying a generic period t, we can focus only on a strategy for a contestant without an all-in look and one for a contestant with an all-in look. The number of contestants with an all-in look in period t is denoted by $\nu_t \in \{0, ..., n\}$; if $\nu_t = 1$, period t is what we call a period with all-in monopoly and if $\nu_t \geq 2$ a period with all-in oligopoly. As all contestants have an all-in look equal to ε in the eyes of their rivals in the first period, $\nu_1 = n$ (if $\varepsilon > 0$) and $\varepsilon_1 = \varepsilon$. Finally, note that, in every period, the pair (ε_t, ν_t) is a sufficient statistic for past play: it contains all the information contestants need to choose their actions.

[10]Section 6 discusses the robustness of our results to alternative choices of all-in efforts. Assuming that all-in automatons are locked into exerting only one effort simplifies the derivation of equilibrium beliefs: any non-all-in effort can only be exerted by a rational contestant. Hence, given that automatons exert effort 1, exerting an effort greater than 1 for a rational contestant is strictly dominated by 0 and thus easily ruled out in equilibrium. Alternatively, one could model all-in automatons as rational but with payoffs different from those of rational contestants in a way that leads them to always choose the all-in effort, though they may choose non-all-in efforts too. With such alternative specification, off-the-equilibrium path beliefs over automatons deviating to non-all-in efforts would have to be carefully analyzed.

[11]For a standard definition, see Definition 3.2 in Fudenberg and Tirole (1991).

[12]See Section 6 for the possibility of regaining an all-in look.

[13]Section 6 discusses asymmetric beliefs over all-in looks and non-type-symmetric equilibria.

Throughout the paper, a benchmark useful to single out the effects of all-in looks is the version of the above model without all-in looks (i.e., $\varepsilon = 0$). In such a benchmark, the equilibrium expected aggregate effort equals 1 and payoffs 0 in every period—see Baye, Kovenock, and De Vries (1996).

4 Illustrative examples

4.1 Two-period example

To gradually build intuition, consider the TSPBE when $T = n = 2$ and the first-period reputation equals $\varepsilon_1 > 0$ for both contestants. We depict the TSPBE strategies in Figure 1 and explain it in what follows.

By backward induction, we start with the second period. For a contestant who faces an opponent with reputation $\varepsilon_2 > 0$, exerting all-in effort 1 (or a larger effort) is strictly dominated by exerting effort zero. Thus, undominated efforts must be non-all-in (smaller than 1) and can only win against a rational opponent (an event with probability $1 - \varepsilon_2$). In other words, a contestant who faces an opponent with reputation $\varepsilon_2 > 0$ is, equivalently, fighting for an "effective prize" of $1-\varepsilon_2$ in a standard all-pay auction. From Baye, Kovenock, and De Vries (1996), we have a complete characterization of equilibrium strategies and payoffs for a two-player standard all-pay auction with symmetric or asymmetric prizes. In particular, applying their characterizations with adjusted effective prizes, our focus on TSPBE implies that three possibilities emerge in the second period:

1. If none of the contestants has reputation, then they fight for an effective prize of 1: hence, contestants mix uniformly on $[0, 1]$ and individual payoffs are zero.

2. If both contestants have reputation (equal to ε_2), then they fight for an effective prize of $1 - \varepsilon_2$: hence, contestants mix uniformly on $[0, 1 - \varepsilon_2]$ and payoffs are zero.[14]

3. If only one contestant has reputation (equal to ε_2), then she fights for an effective prize of 1 and her opponent fights for $1 - \varepsilon_2$: hence, the contestant with reputation mixes uniformly on $[0, 1 - \varepsilon_2]$ and the one without reputation mixes uniformly on $[0, 1 - \varepsilon_2]$ with probability (w.p.) $1 - \varepsilon_2$ and exerts zero effort w.p. ε_2. The payoff of the former contestant is ε_2 and that of the latter zero.[15]

[14] These equilibrium features are reminiscent of the one-shot all-pay auction with binary and symmetric private information studied in Konrad (2004). Indeed, the equilibrium behavior of a rational player is a special case of that of an altruistic player in Konrad (2004)'s Proposition 1, with $\gamma = 1 - \varepsilon_2$ and $V_A = 1$ applied to his Equation (6). Such special case holds, in general, only for the very last period $t = T$.

[15] These equilibrium features are reminiscent of the one-shot all-pay auction with binary and asymmetric

As for the first period, contestants anticipate that the only positive future payoff occurs, as discussed above, if only one of the two contestants reaches the second period with reputation (by having exerted all-in effort 1 in the first period). Thus, efforts below and sufficiently close to 1 are never played in equilibrium because strictly dominated by 1, which entails the prospect of future reputational benefits. On the other hand, non-all-in efforts entail a loss of reputation and of future benefits; standard reasoning for all-pay auctions between two identical contestants imply that non-all-in efforts are uniformly distributed on an interval with lower bound at 0 (so that contestants' cumulative equilibrium payoffs are 0). Its upper bound is strictly below 1 for the above reasoning. Therefore, contestants' strategies can contain, at most: (1) a mass point at 1, and (2) a uniform distribution over lower efforts.

The uniform distribution is a necessary part of the equilibrium. In fact, if contestants exerted all-in effort with certainty, then they would tie in the first period and obtain a negative expected payoff, and their second period expected payoff would be 0. Hence, a pure strategy with all-in effort cannot be sustained in equilibrium. But is the mass point at 1 part of the equilibrium strategy?

If ε is large, the answer is No. In fact, the costly all-in effort does not pay off because of the high risk of an all-in automaton opponent. More formally, if $\varepsilon \to 1$, then the first-period payoff of exerting an all-in effort tends to $-1/2$ (certainty of a tie) and the second-period expected payoff tends to 0 because of the almost certainty of being up against an all-in automaton. If instead ε is sufficiently low, the answer is Yes. Suppose an equilibrium existed with no mass point at 1, and consider a deviation to 1. This deviation would give cumulative payoff

$$(1-\varepsilon)\cdot 2+\varepsilon\cdot\left(\frac{1}{2}\right)-1=1-\frac{3}{2}\varepsilon. \tag{1}$$

The 2 that multiplies the probability of a rational opponent arises because against this opponent a bid of 1 wins for sure in the first period, and in the second period the opponent would be certain of facing an all-in automaton and hence exert zero effort, thus leaving the deviating contestant with a payoff of 1 in the final period. The 1/2 that multiplies the probability of an all-in automaton opponent arises because against this opponent a bid of 1 ties in the first period, and in the final period the deviator would be certain of facing an all-in automaton. Therefore, from (1), if $\varepsilon < 2/3$, an equilibrium without mass point at 1 cannot be sustained, because deviating to 1 would be profitable.

The upper bound of the support of the uniform distribution and the size of the mass

private information studied in Szech (2011). The equilibrium where $\varepsilon_2 = 1/2$ is a special case of the equilibrium strategy of a "Weak" player 1 or 2 characterized in part 3 of Szech (2011)'s Proposition 1, with $p_1 = 1/2$, $p_2 = 0$, $C_1 = C_2 = 1$, and $c_1 = c_2 = c$ with $c \downarrow 0$. Such special case holds, in general, only for the very last period $t = T$.

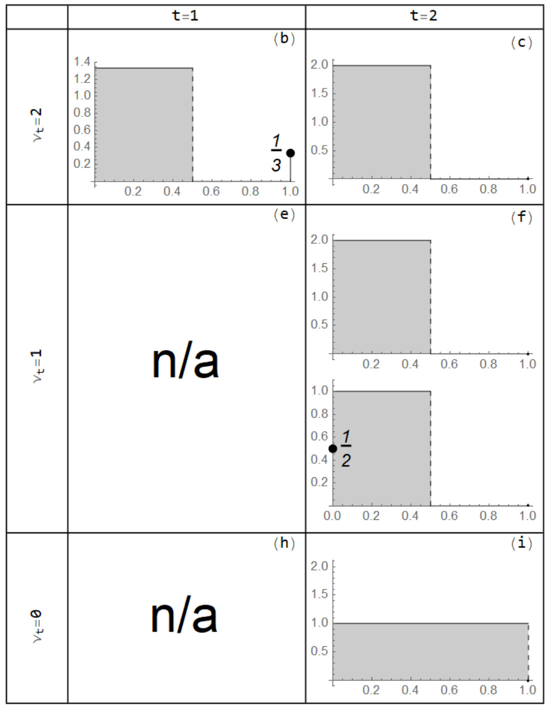

Figure 1: Unique TSPBE strategies when $\{T, n, \varepsilon\} = \{2, 2, 1/4\}$. The figure depicts probability density functions of equilibrium efforts. In panel (f), the top (bottom) part depicts the behavior of the contestant with (without) an all-in look; e.g., in panel (b), contestants fight all-in w.p. 1/3, and exert non-all-in effort (uniformly distributed on $[0, 1/2]$) otherwise.

point at 1 are endogenously determined so that the payoff of any effort in the support of the mixed strategy is 0. For instance, for $T = n = 2$ and $\varepsilon_1 = 1/4$, Figure 1 shows the unique equilibrium strategies. From panel (b), three effort levels yield the same cumulative payoff:

1. A **negligible effort** yields:

    (a) zero current-period payoff, as a negligible effort never wins in panel (b), and

    (b) zero future payoff, as a negligible effort entails loss of reputation, and hence loss of the opportunity of monopolizing reputation and having a positive future payoff.

2. An **effort equal to** 1/2 yields:

    (a) zero current-period payoff, equal to $(3/4)\,(2/3) - 1/2$, as an effort equal to 1/2 entails victory only if the rival is rational and exerts non-all-in effort, and

    (b) zero future payoff, as an effort equal to 1/2 entails loss of reputation.

3. The **all-in effort** yields:

   (a) a negative current-period payoff of $-1/4$: that is,

   $$(3/4)\,(2/3) + [(3/4)\,(1/3) + (1/4)]\,(1/2) - 1,$$

   as an all-in effort entails victory only if the rival is rational and exerts non-all-in effort, or the rival exerts all-in effort (w.p. given in the square-bracketed term of the above-displayed expression) but loses the tie-break,

   (b) a positive expected future payoff of $1/4$, as we now explain. First, as all-in efforts are exerted w.p. $1/3$ in $t = 1$, then in $t = 2$ Bayes' rule and $\varepsilon_1 = 1/4$ imply that $\varepsilon_2 = \frac{1/4}{1/4+(3/4)(1/3)} = \frac{1}{2}$. Second, recalling that the second-period payoff of the monopolist equals ε_2, then the expected payoff equals $(1/2)\,(3/4)\,(2/3) = 1/4$ as the second-period payoff of the monopolist only occurs if her rival is rational and lost reputation in the first period.

One can easily verify that any effort in $[0, 1/2]$ also yields 0 payoff. This explains the first-period TSPBE strategy depicted in Figure 1.

4.2 Three-period example

We now extend the example in Section 4.1 to $T = 3$ and depict the TSPBE strategies in Figure 2, so as to highlight key *dynamics* and allow for the possibility that reputation is monopolized in a non-terminal period. As for TSPBE beliefs, for simplicity, we now set $\varepsilon = 1/8$, while in Figure 1 we had $\varepsilon = 1/4$, so as to *re*-use panels (b), (c), (f), and (i) of Figure 1 in our three-period example of Figure 2 without changes. In fact, in the unique equilibrium of the game with $\{T, n, \varepsilon\} = \{3, 2, 1/8\}$, any player who enters the second period with reputation is believed by others to be an all-in automaton w.p. $\varepsilon_2 = 1/4$ by Bayes rules, corresponding to the prior probability of an all-in automaton in the example of Figure 1.

All-in efforts are never chosen in the very last period (as building an all-in look gives no future benefits) or by a contestant who has already lost her reputation (as an all-in look cannot be restored once given up). Hence, along the equilibrium path, ν_t decreases over time. Moreover, ε_t increases over time—as long as some all-in looks are maintained by someone—as the fear of each other increases as more and more all-in efforts are observed: $\varepsilon_1 = 1/8$, $\varepsilon_2 = 1/4$, and $\varepsilon_3 = 1/2$. And as long as both players keep fighting all-in, each suffers negative per-period payoff in the hope that her all-in look outlasts that of the rival

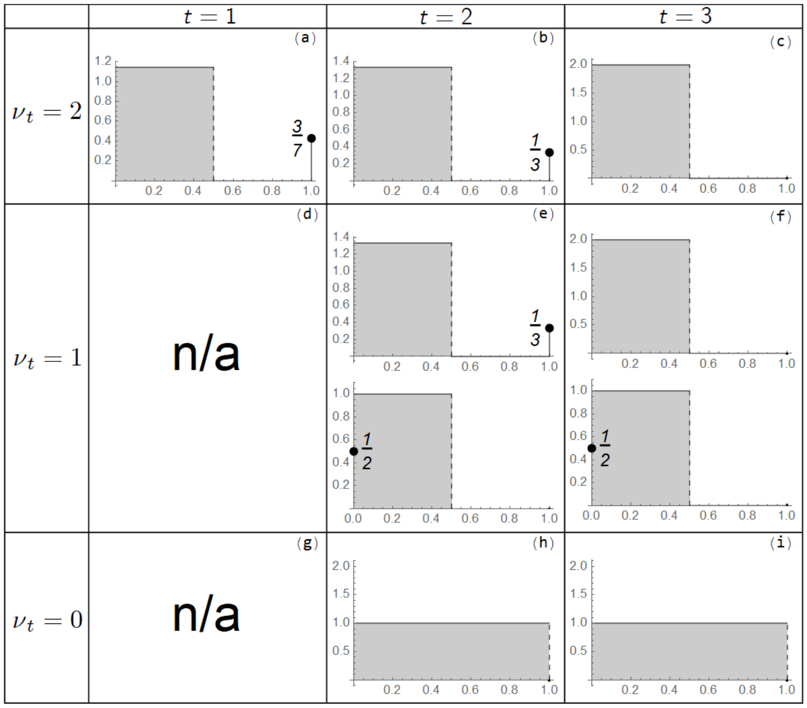

Figure 2: Unique TSPBE strategies when $\{T, n, \varepsilon\} = \{3, 2, 1/8\}$. The top (bottom) of panels (e) and (f) depicts the behavior of the contestant with (without) an all-in look; e.g., in panel (e), the contestant without all-in look exerts zero effort w.p. 1/2, and exerts non-all-in effort (uniformly distributed on $[0, 1/2]$) otherwise.

so as to cash in on her reputational monopoly.[16]

A key feature that differentiates the two- and the three-period examples is that, in the latter, there is an intermediate period $t = 2$ where beliefs can be asymmetric (contrarily to the first period of the game) and the future still counts (contrarily to the last period of the game): see panel (e). Here, the all-in monopolist mixes between fighting all-in and a continuum of non-all-in fighting efforts: in fact,

1. if she fights all-in, she has a current-period payoff of 0 but a positive future payoff (equal to 1/2, which is ε_3: the third-period monopolist's payoff) by maintaining—and actually boosting—her all-in look, and

2. if she fights non-all-in, she has a future payoff of 0 because she loses her all-in look, but a positive current-period payoff (equal to 1/2, as a negligible effort has zero cost and 1/2 win probability).

[16]Propositions 3 and 7–9 establish that these qualitative properties hold more generally, in the (T, n, ε)-game.

Thus, her cumulative payoff is 1/2 both by fighting all-in and not, and she is indifferent between boosting her all-in look for tomorrow, or cashing in on her all-in look today.[17]

Another key feature that differentiates the two- and the three-period examples is that it is now possible that the per-period expected aggregate effort exceeds 1: in particular, this happens in panel (a). The fact that the per-period expected aggregate effort exceeds 1 is possible only in sufficiently early periods; indeed, in all panels other than (a), aggregate effort never exceeds 1. Aggregate effort may exceed 1 in an all-in oligopoly because contestants fight not only for the per-period prize of 1 but also for outlasting the rivals' all-in looks. In an all-in monopoly, the per-period aggregate effort is strictly below 1, because contestants fight for a per-period prize of value 1 and there is no competition for outlasting the rivals' all-in looks. One can say that all-in looks, when monopolized, *discourage* efforts: the expected aggregate effort in an all-in monopoly is lower than in the natural benchmark without all-in looks at all ($\varepsilon = 0$). In contrast, in an all-in oligopoly, the expected aggregate effort may be higher than in the benchmark without all-in looks; thus, all-in looks may *encourage* efforts.[18]

4.3 Beyond the above examples

The two examples discussed above capture many of the key forces underlying the equilibrium of the general model. But some of the results in the two examples do not hold generally. We conclude this section with an intuitive description of the main changes that arise when $T > 3$, $n > 2$, and ε takes an arbitrary value.

If $T > 3$, an important difference emerges between an oligopoly and a monopoly: while an all-in monopoly can be sustained indefinitely (that is, until T, no matter how large T is), an all-in oligopoly has a maximum duration and may end well before the very last period. In fact, in an oligopoly, as reciprocal beliefs increase over time, so does an oligopolist' fear of being up against an actual automaton. This fear eventually deters contestants from fighting all-in. In general, Proposition 7 characterizes the maximum duration of an all-in oligopoly for every initial belief ε. However, Proposition 4 shows that a similar maximum duration does not exist for an all-in monopoly; the monopolist may keep her all-in look until period T because she never fears being up against an all-in automaton.

If $n > 2$, the contestants without all-in looks are inactive (exert zero effort with certainty)

[17]Note that, when a monopolist is in a non-terminal period as in panel (e), we can neither sustain an equilibrium with only all-in efforts by the monopolist, nor one with only non-all-in efforts by the monopolist. The former would imply that the contestant without reputation best replies with certainty of zero effort, which in turn makes all-in efforts dominated by negligible efforts for the monopolist. The latter implies that the monopolist profitably deviates to an all-in effort, so as to make the contestant without reputation certain of her all-in type, by Bayes' rule, and hence win with negligible effort in the subsequent period.

[18]Corollary 2 and propositions 5 and 8 establish that these qualitative properties hold more generally, in the (T, n, ε)-game.

in a period when at least two contestants have an all-in look: the fierce competition among the contestants with an all-in look to become the all-in monopolist deters the contestants without an all-in look from exerting effort at all (see Proposition 3). This endogenous participation that emerges in the equilibrium of the game with more than three contestants resembles the long-standing literature finding that reputation-building deters future entry of rivals (see, for instance, Mailath and Samuelson, 2006). Furthermore, as shown in Figure 2 for $n = 2$, the upper bound of the support of non-all-in efforts is $1/2$ whenever at least one contestant has an all-in look—that is, in all panels (a)-(f). In the general (T, n, ε)-game, such upper bound may vary with ν_t and t.[19]

If $\varepsilon < 1/8$, as ε decreases and approaches 0, the equilibrium behavior approaches a uniform on $[0, 1]$ in all panels. If instead $\varepsilon > 1/8$, then fighting all-in could now disappear from the equilibrium of both panels (a) and (b) because of the high fear of being up against an automaton.[20] As long as panel (e) can be reached (i.e., $\varepsilon < 2/3$), the all-in monopolist always fights all-in with strictly positive probability—see Proposition 2.

5 Equilibrium

This section fully characterizes and analyzes the unique TSPBE of the general (T, n, ε)-game and highlights the structural equilibrium differences between periods when zero, one, or multiple contestants have an all-in look.

5.1 Characterization

We define π_t^{ν} ($\pi_t^{-\nu}$) as the expected cumulative equilibrium payoff of a contestant with (without) an all-in look in period t, calculated as the sum of all per–period equilibrium payoffs from t to T. In what follows, we consider separately the cases when, in a general period t, zero, one, or multiple contestants have an all-in look: respectively, $\nu_t = 0$, $\nu_t = 1$, or $\nu_t \geq 2$. Recall that $\nu_1 = n$ (when $\varepsilon > 0$), so that at $t = 1$ we start with an oligopoly.

The case of $\nu_t = 0$ is equivalent to a standard complete information all-pay auction.

Proposition 1 (Baye, Kovenock, and De Vries (1996)) *In the unique TSPBE, strategies in period $t \in \{2, ..., T\}$ with $\nu_t = 0$ are as follows. Contestants exert effort on $[0, 1]$ with cumulative density function (CDF) $x^{\frac{1}{n-1}}$. Also, $\pi_t^{-\nu} = 0$.*

[19] Such upper bound decreases in ε_t and p_t (the probability a contestant with an all-in look exerts all-in effort), and ε_t increases while p_t decreases over time (Proposition 3). If $n = 2$, these two effects balance out.

[20] For panel (a), $\varepsilon > 2/3$ guarantees that fighting all-in is not part of the equilibrium (so that panel (b) is never reached)—see Proposition 3. For panel (b), one can show that if $\varepsilon > 0.233$, then $\varepsilon_2 > 0.414$ and fighting all-in is not part of the equilibrium in panel (b).

For a period t with an **all-in monopoly**, equilibrium strategies and payoffs are as follows.

Proposition 2 *Let*

$$p_t = \frac{\varepsilon_t^{\frac{1}{T-t+1}} - \varepsilon_t}{1-\varepsilon_t} \text{ and } q_t = \varepsilon_t^{\frac{1}{(T-t+1)(n-1)}}$$

and consider the following two CDFs on $x \in \left[0, 1-q_t^{n-1}\right]$*:*

$$F_t(x) = \frac{x}{(1-\varepsilon_t)(1-p_t)\left(x+q_t^{n-1}\right)^{\frac{n-2}{n-1}}} \text{ and } G_t(x) = \frac{\left(x+q_t^{n-1}\right)^{\frac{1}{n-1}} - q_t}{(1-q_t)}.$$

In the unique TSPBE, strategies in period $t \in \{2,...,T\}$ *with* $\nu_t = 1$ *are as follows. The effort of the contestant with an all-in look follows* F_t *w.p.* $1-p_t$*, and is* 1 *w.p.* p_t*. The effort of each contestant without an all-in look is* 0 *w.p.* q_t*, and follows* G_t *w.p.* $1-q_t$*. Also,* $\pi_t^{\nu} = \varepsilon_t^{\frac{1}{T-t+1}}$ *and* $\pi_t^{-\nu} = 0$*.*

We now provide the intuition for the equilibrium quantities in Proposition 2. In a non-terminal period, the all-in monopolist mixes between (1) fighting all-in and boosting her all-in look tomorrow, and (2) a continuum of non-all-in fighting efforts (distributed according to F_t) to cash in on her all-in look today.[21] Indeed, the all-in monopolist who exerts non-all-in effort gives up her all-in look, and thus we can focus exclusively on the current-period payoff; for an effort $x \in \left(0, 1-q_t^{n-1}\right]$, she obtains $(q_t + (1-q_t) G_t(x))^{n-1} - x$ when the $n-1$ rivals use the strategy in Proposition 2. Also, if she exerts an arbitrarily small, but strictly positive effort, she obtains q_t^{n-1}. Hence, in equilibrium, we must have the following

$$(q_t + (1-q_t) G_t(x))^{n-1} - x = q_t^{n-1}. \tag{2}$$

Exerting effort $x \in \left(1-q_t^{n-1}, 1\right)$ or $x = 0$ is not a profitable deviation. Exerting all-in effort $x = 1$ yields a zero current-period payoff (as the monopolist wins with certainty) and an endogenously-determined cumulative equilibrium payoff tomorrow, which depends on others' updated beliefs about the monopolist's all-in look. Calculating this payoff requires a non-trivial recursive characterization (see the Proof of Proposition 2 for details) which eventually

[21] In the equilibrium of a non-terminal period, a monopolist must play with positive probability *both* (1) fighting all-in, and (2) a continuum of non-all-in fighting efforts. In fact, if the monopolist were to fight all-in with certainty—only (1)—then rivals would remain inactive with certainty and the monopolist would deviate to an arbitrarily small but strictly positive effort today and obtain a payoff of 1, which is greater than the payoff $\pi_{t+1}^{\nu_t}$ the monopolist would obtain in the next period. If the monopolist were to fight non-all-in with certainty—only (2)—then fighting all-in would make the rival certain that she is an all-in automaton and thus be a profitable deviation as it fully discourages rivals.

yields $\pi_t^\nu = \varepsilon_t^{\frac{1}{T-t+1}}$. As a monopolist must be indifferent between non-all-in and all-in efforts, π_t^ν must also equal q_t^{n-1} (see (2)); this pins down the equilibrium value of q_t.

A contestant without an all-in look exerting effort $x \in \left[0, 1 - q_t^{n-1}\right]$ obtains

$$(1-\varepsilon_t)(1-p_t) F_t(x) (q_t + (1-q_t) G_t(x))^{n-2} - x = 0, \tag{3}$$

as she wins only if the all-in monopolist is rational and exerts non-all-in effort, and if she exerts effort greater than that of the monopolist and the $n-2$ contestants without all-in looks. As the support of F_t and G_t are the same, $F_t\left(1 - q_t^{n-1}\right) = 1$; this pins down the equilibrium value of p_t.

Importantly, note that a contestant who achieves an all-in monopoly through a series of all-in fighting efforts can cash in on her all-in monopoly (and obtain a strictly positive payoff) by exerting non-all-in effort. This can be seen considering an all-in monopolist who exerts a strictly positive, but arbitrarily small effort; if all her rivals remain inactive (which happens in equilibrium w.p. q_t^{n-1}), she wins 1 with a negligible effort, and thus she enjoys a strictly positive per-period payoff.

Note that the payoff of the monopolist equals $\varepsilon_t^{\frac{1}{T-t+1}}$ in period t, and hence it increases in both ε_t and T. The former is due to the beneficial effect for the monopolist of having her rivals believe that she is all-in with high probability. The latter is due to the beneficial effect of the longer time horizon over which a monopolist can enjoy her reputational monopoly.

An interesting comparison, at this point, is between the one-sided reputation in Kwiek (2011)'s repeated second-price auctions and our all-in monopoly in repeated all-pay auctions. In the former, "with two equally and very patient bidders, then the bidder with reputation collects almost the entire surplus from this repeated auction–as if there were no other bidder at all" (Kwiek, 2011; p. 988). In the latter, as $T \to \infty$ (in order to "match" our finite horizon game with Kwiek's infinite horizon one), the monopolist's cumulative equilibrium payoff tends to 1 (see Proposition 2), as if there were no other bidder in the current period. But, as soon as the monopolist cashes in her reputation, per-period payoffs are 0.

For a period t with an **all-in oligopoly**, Proposition 3 below characterizes the strategies and payoffs in the TSPBE. This equilibrium is structurally different from the one in Proposition 2. The competition between contestants with an all-in look drags down their individual cumulative equilibrium payoff to 0 (and discourages contestants without all-in looks, who remain inactive). Such 0 payoff, letting p_t be the probability that a contestant with an all-in look fights all-in, implies,

$$\begin{aligned}1 = \;& \textstyle\sum_{l=0}^{\nu_t-1} \frac{1}{l+1} \begin{pmatrix} \nu_t - 1 \\ l \end{pmatrix} (\varepsilon_t + (1-\varepsilon_t)\, p_t)^l \, (1 - (\varepsilon_t + (1-\varepsilon_t)\, p_t))^{\nu_t - 1 - l} \\ & + (1 - (\varepsilon_t + p_t(1-\varepsilon_t)))^{\nu_t - 1} \left(\tfrac{\varepsilon_t}{\varepsilon_t + p_t(1-\varepsilon_t)}\right)^{\frac{1}{T-t}} .\end{aligned} \tag{4}$$

The right-hand side (RHS) of (4) is the benefit of fighting all-in, and the left-hand side (LHS) is its cost. The benefit has two components. The first component is the summation that captures the expected share of the current period's prize, taking into account that there can be any number between 0 and $\nu_t - 1$ of rivals with an all-in look (rational or all-in automatons) who may fight all-in and tie; in fact, the probability that an individual rival with an all-in look exerts all-in effort is $\varepsilon_t + (1-\varepsilon_t)\, p_t$. The second component (second line of (4)) has first the probability that all the other $\nu_t - 1$ contestants with an all-in look exert non-all-in efforts today, and second the benefit of being an all-in monopolist tomorrow, which equals $\varepsilon_{t+1}^{1/(T-t)}$ from Proposition 2. In (4), by Bayes' rule, ε_{t+1} is a function of ε_t and p_t.[22]

Condition (4) is key in characterizing the equilibrium for a period t with an all-in oligopoly. Whether (4) has a solution with $p_t \geq 0$ or not crucially depends on whether ε_t is small enough; intuitively, if ε_t was arbitrarily close to 1, fear of all-in automaton rivals would be high enough to guarantee that a rational contestant would never exert all-in effort. The upper bound that guarantees existence of an equilibrium with $p_t \geq 0$ is denoted by $\bar{\varepsilon}_t(\nu_t)$ and it is the unique solution to (4) with $p_t = 0$, which reads[23]

$$1 = \frac{1}{\nu_t} \sum_{l=0}^{\nu_t - 1} (1-\varepsilon_t)^l + (1-\varepsilon_t)^{\nu_t - 1}. \tag{5}$$

A straightfoward analysis of (5) gives the following.

Lemma 1 $\bar{\varepsilon}_t(\nu_t)$ *exists, is unique, smaller than* $2/3$*, and strictly decreasing in* ν_t*.*

Intuitively, the larger the number ν_t of contestants with an all-in look, the less profitable it is to exert all-in effort, the smaller is the region of ε_t for which an equilibrium with all-in effort can be sustained.

We are now ready to provide the characterization of the TSPBE in a period t with an all-in oligopoly.

[22] Bayes' rule reads $\varepsilon_{t+1} = \varepsilon_t / (\varepsilon_t + (1-\varepsilon_t)\, p_t)$.

[23] For more details, see the Proof of Proposition 3.

Proposition 3 *Consider the following CDF:*

$$F_t(x) = \frac{x^{\frac{1}{\nu_t - 1}}}{(1-\varepsilon_t)(1-p_t)} \text{ if } x \in \left[0, ((1-\varepsilon_t)(1-p_t))^{\nu_t - 1}\right].$$

In unique TSPBE, in period $t \in \{1, ..., T\}$ with $\nu_t \geq 2$, $\pi_t^{\nu} = \pi_t^{-\nu} = 0$ and strategies are as follows. The effort of the $n - \nu_t$ contestants without all-in looks is 0, and

- *if $t = T$ or if $t < T$ and $\varepsilon_t \in [\bar{\varepsilon}_t(\nu_t), 1]$, the effort of each of the ν_t contestants with all-in looks follows F_t with $p_t = 0$,*
- *if $t < T$ and $\varepsilon_t \in [0, \bar{\varepsilon}_t(\nu_t)]$, the effort of each of the ν_t contestants with all-in looks follows F_t w.p. $1 - p_t$ and is 1 w.p. p_t, where p_t is the unique solution implicitly defined by (4) for given ε_t. Finally, $p_t \in [0, 2/3]$.*

Note that F_t in Proposition 3 has a simple interpretation, similar to that for G_t of Proposition 2. A contestant with an all-in look who exerts effort $x \in \left[0, ((1-\varepsilon_t)(1-p_t))^{\nu_t - 1}\right]$ when the rivals use the strategy described in Proposition 3 for $\varepsilon_t \in (0, \bar{\varepsilon}_t(\nu_t)]$ gives up her all-in look and obtains $((1-\varepsilon_t)(1-p_t))^{\nu_t - 1} F_t(x)^{\nu_t - 1} - x$. Also, she must be indifferent between such an effort and an arbitrarily small effort that yields a 0-payoff (as she loses with certainty to the other contestants with all-in looks). Such indifference explains the equilibrium value of F_t in Proposition 3.[24] Note as well that Proposition 3 implies that the overall expected aggregate effort in the game equals the overall value of all the prizes as $\pi_1^{\nu} = 0$.

An interesting comparison, at this point, is between the two-sided reputation in Kwiek (2011)'s repeated second-price auctions and our all-in oligopoly in repeated all-pay auctions. In the former, a war of attrition emerges, with initial periods of particularly high bids, and the "competition is so dramatic that in effect both players get the lowest conceivable payoff" (Kwiek, 2011; p. 990). The exact same happens in the latter.

5.2 All-in Monopoly: Further Properties and Applications

This section describes important properties of the equilibrium under all-in monopoly other than those already characterized in Proposition 2. Proposition 4 shows that, starting in any period t with an all-in monopoly and for any belief level $\varepsilon_t > 0$, $\nu_T = 1$ will occur with strictly positive probability even if the last period T is arbitrarily far ahead in the future: a monopoly may last indefinitely—that is, until $T < \infty$, no matter how large T is. Finally, Proposition 4 also characterizes the law of motion of beliefs over time.

[24] It is not a profitable deviation for a contestant with an all-in look to exert effort $x \in \left(((1-\varepsilon_t)(1-p_t))^{\nu_t - 1}, 1\right)$.

Proposition 4 *In the unique TSPBE, in period $t \in \{2, ..., T\}$ with $\nu_t = 1$,*

1. *$\forall \varepsilon_t > 0$, $\nu_T = 1$ occurs with strictly positive probability,*
2. *if $\nu_{t+1} = 1$, then $\varepsilon_{t+1} = \varepsilon_t^{\frac{T-t}{T-t+1}}$. Also, $\varepsilon_{t+1} > \varepsilon_t$.*

The intuition for the first result is that the all-in monopolist does not fear being up against all-in automatons and hence she always plays all-in effort with strictly positive probability (see Proposition 2). The second result follows by Bayes' rule and Proposition 2.

An all-in monopoly never yields per-period expected aggregate effort larger than 1 (the per-period prize), as the following proposition shows.[25]

Proposition 5 *In the unique TSPBE, in period $t \in \{2, ..., T\}$ with $\nu_t = 1$, the expected aggregate effort in period t is lower than 1.*

There are two forces behind Proposition 5. When $\varepsilon_t = 0$, there is common-knowledge of rationality among contestants and hence the expected aggregate effort in period t is 1 (recall Proposition 1). When $\varepsilon_t > 0$, the all-in monopolist has an incentive to fight all-in to increase ε_{t+1}: pretending to be an all-in automaton becomes credible and profitable. This force tends to increase the expected aggregate effort. However, when $\varepsilon_t > 0$, the expected effort of the contestants without an all-in look decreases because of the increased fear that the all-in monopolist is an actual automaton. Proposition 5 shows that the latter effect wins out: the discouragement of contestants without all-in looks dominates the monopolist's incentive to fight all-in and per-period expected aggregate effort never exceeds 1 in an all-in monopoly.

A natural benchmark is a period when no contestant has an all-in look ($\nu_t = 0$). Here, the expected aggregate effort is 1 (see Proposition 1). Proposition 5 shows that the introduction of all-in looks discourages efforts in an all-in monopoly ($\nu_t = 1$). One can say that all-in looks *discourage* efforts in repeated contests: the expected aggregate effort in a period of all-in monopoly is lower than in the natural benchmark without all-in looks ($\varepsilon_t = 0$).

We characterize the per-period win probability of the all-in monopolist.

Proposition 6 *In the unique TSPBE, in period $t \in \{2, ..., T\}$ with $\nu_t = 1$, the contestant with an all-in look wins in period t w.p.*

$$\frac{1}{n} + \frac{(n-1)^2}{n} \frac{\varepsilon_t^{\frac{1}{T-t+1}} - \left(\varepsilon_t^{\frac{1}{T-t+1}}\right)^{\frac{n}{n-1}}}{1 - \varepsilon_t^{\frac{1}{T-t+1}}}. \tag{6}$$

[25] Proposition 5 considers only the efforts of rational players. The conclusion that the expected aggregate effort is lower than 1 would however carry over if we consider the expected effort of a player who may be rational or all-in with interior probabilities, because the effort of the all-in player is always 1.

The expression in (6) is greater than $1/n$; the all-in monopolist is more likely to win than any contestant without an all-in look. And finally, one can use (6) to show the following.

Corollary 1 *In the unique TSPBE, in period $t \in \{2, ..., T\}$ with $\nu_t = 1$, the contestant with an all-in look wins with a probability that is increasing in ε_t and decreasing in n and t.*

The win probability of the all-in monopolist in period t increases in ε_t because contestants without an all-in look are discouraged by higher beliefs over the monopolist's all-in look. The win probability of the all-in monopolist also intuitively decreases in the number of rivals. It also decreases in t; for a fixed belief over the monopolist's all-in look, the earlier she achieves such a belief level, the larger her win probability.

5.3 All-in Oligopoly: Further Properties and Applications

This section describes important properties of the equilibrium in an all-in oligopoly other than those already characterized in Proposition 3. Proposition 7 shows that, in any period t with an all-in oligopoly and for any belief level $\varepsilon_t > 0$, the number of periods for which multiple contestants can maintain all-in looks does not cover the entire remaining duration of the game (that is, till period T) if T is large enough. Finally, Proposition 7 also characterizes the law of motion of beliefs on all-in looks over time.

Proposition 7 *In the unique TSPBE, in period $t \in \{1, ..., T\}$ with $\nu_t \geq 2$,*

1. *$\forall \varepsilon_t > 0$, $\nu_\tau \geq 2$ occurs w.p. 0 if*

$$\tau \geq t + \frac{\log\left(\frac{1}{2}\frac{\varepsilon_t}{1-\varepsilon_t}\right)}{\log\left(\frac{2}{3}\right)}. \tag{7}$$

2. *If $\varepsilon_t \in (0, \bar{\varepsilon}_t(\nu_t)]$ and $\nu_{t+1} \geq 1$, then*

$$1 - \frac{\varepsilon_{t+1}}{\nu_t \varepsilon_t}\left(1 - \left(1 - \frac{\varepsilon_t}{\varepsilon_{t+1}}\right)^{\nu_t}\right) = \left(1 - \frac{\varepsilon_t}{\varepsilon_{t+1}}\right)^{\nu_t - 1} \varepsilon_{t+1}^{\frac{1}{T-t}}, \tag{8}$$

and $\varepsilon_{t+1} > \varepsilon_t$.

The upper bound on the number of periods with an all-in oligopoly in (7) arises because, over time, fighting all-in increases ε_t, i.e., the belief level about all-in automatons. Furthermore, since we know that $p_t \leq 2/3$ by Proposition 3, Bayes' rule (see Footnote 22) gives $\varepsilon_{t+1} > \varepsilon_t / (\varepsilon_t + 2(1-\varepsilon_t)/3)$, so that the belief follows an increasing sequence that converges to 1. And we know from Proposition 3 that fighting all-in becomes too costly for ε_t large enough ($\varepsilon_t > \bar{\varepsilon}_t(\nu_t) \in [0, 2/3]$) because of the fear of being up against actual all-in automatons. Hence, the all-in oligopoly must end with certainty before period T, if T is sufficiently large. Note also that, for any fixed ε_1, the upper bound on the number of periods with an all-in oligopoly remains finite even if $T \to \infty$.

An all-in oligopoly may yield a per-period expected aggregate effort that exceeds the prize at stake in that period if ε_t is small, as the following proposition shows.[26]

Proposition 8 *In the unique TSPBE, in period $t \in \{1, ..., T-1\}$ with $\nu_t \geq 2$, $\exists \hat{\varepsilon}_t > 0$ such that, $\forall \varepsilon_t \in (0, \hat{\varepsilon}_t)$, the expected aggregate effort in period t is strictly greater than 1.*

The intuition behind Proposition 8 is as follows. When $\varepsilon_t = 0$, the expected aggregate effort in period t is 1. When $\varepsilon_t > 0$, the contestants with an all-in look have an incentive to fight all-in in order to pretend to be an all-in automaton and increase ε_{t+1}. The contestants without an all-in look remain inactive. Thus, for sufficiently low ε_t, per-period expected aggregate effort exceeds 1: contestants with an all-in look fight particularly hard not only for the per-period prize of 1, but also for the prospect of achieving a future all-in monopoly. To see why this result requires ε_t small enough, consider the extreme case of ε_t close to 1; a contestant with an all-in look is almost certain of being up against $\nu_t - 1$ all-in automatons, and hence any strictly positive effort is costly and yields a negligible win probability.

As in the natural benchmark without all-in looks ($\varepsilon_t = 0$) the expected aggregate effort is 1, Proposition 8 shows that the introduction of all-in looks may or may not encourage efforts in an all-in oligopoly ($\nu_t \geq 2$), according to the belief level ε_t. All-in looks *encourage* efforts in repeated contests if ε_t is small enough: the expected aggregate effort may be higher than in the benchmark without all-in looks.

Building on Proposition 7 and Proposition 8, we obtain that per-period expected aggregate effort may exceed 1 only in sufficiently early contests, as the next result shows.

Corollary 2 *In the unique TSPBE, in period $t \in \{2, ..., T\}$, $\exists \bar{t} \in \{2, .., T-1\}$ s.t. per-period expected aggregate effort is less than 1 if $t > \bar{t}$.*

Under all-in oligopoly, each contestant with an all-in look has a per-period win probability equal to $1/\nu_t$ because contestants without an all-in look are inactive and the equilibrium we consider is type-symmetric.

Proposition 9 *In the unique TSPBE, in period $t \in \{1, ..., T\}$ with $\nu_t \geq 2$, each contestant with an all-in look wins w.p. $1/\nu_t$.*

[26] An analytical representation of expected aggregate effort under all-in oligopoly is, in general, not tractable as (4) is a polynomial of degree up to n.

6 Extensions and Robustness

Our framework is intentionally parsimonious and thus it is natural to discuss its limitations and some possible extensions. In particular, the tractability that allowed us to analyze the general (T, n, ε)-game (with $2 \leq T < \infty$, $n \geq 2$, and any ε) is due to the interplay of five simplifying assumptions: the ex-ante symmetry between contestants, the impossibility of regaining an all-in look, the focus on type-symmetric equilibria, the all-in effort equal to 1, and the absence of discounting. In this section, we briefly discuss the prospects for extending our model along each of these five dimensions separately.

Asymmetries among contestants. We considered ex-ante identical contestants who, in equilibrium, enter a period either with the *same* positive all-in look $\varepsilon_t > 0$ or with no all-in look at all. While convenient, having only two possible belief levels in each period for all contestants may be undesirably restrictive. Introducing in the model ex-ante asymmetries that allow contestants to enter a certain period with different levels of (positive) beliefs severely jeopardizes the tractability of our framework. However, there are alternative, tractable ways to introduce ex-ante asymmetries in the model. For instance, our analysis readily carries over if only a subset of contestants has the chance of building an all-in look from the outset. This case is relevant to the applications in the Introduction: organized criminal groups might be about to disband or relocate elsewhere (or a lawyer may be about to retire), and thus they might only be interested in winning the current fight (trial) rather than in building a reputation for toughness (litigiousness). Such an ex-ante asymmetric setup can be captured with a simple adjustment of our framework: out of the n contestants, $l \in \{1, ..., n-1\}$ are long-term contestants who fight in every period (from 1 to T), while the other $n-l$ are short-term contestants, who are replaced in every period by new short-term contestants who observe the entire history of efforts. Then, only long-term contestants may build an all-in look, while the short-term contestants never fight all-in with strictly positive probability as it would result in a negative current-period payoff and no future benefit. Our equilibrium characterization in Section 5 applies to such an $l-$sided model with simple adjustments.[27] We set up our base model with multiple contestants that can build and maintain a reputation, to fit the applications we want to capture: for instance, in turf wars among organized criminal groups fighting over the control of illegal markets, we often see more than one criminal group trying to build a reputation for toughness. Nonetheless, the special case of $l = 1$ is of particular relevance because it speaks to the vast literature on reputation which typically assumes that

[27] The $n-l$ short-term contestants play as if they were long-term contestants who lost their all-in look. Hence, long-term contestants who lost their all-in look and short-term contestants jointly form the set of the $n - \nu_t$ contestants without an all-in look. Our equilibrium characterization in Section 5.1 applies to such an $l-$sided model.

one player (the long-term player) tries to gain reputation in the eyes of short-term players in order to deter future entry. A result similar to our Proposition 2 would apply because, in our model, when a reputational monopoly arises, players without reputation have zero future payoffs and thus behave as if the future would be unaffected by the current actions, so it might as well not exist. Note that, from Proposition 2, a higher reputation ε_t implies that short-term players are inactive with higher probability, in line with the classic deterrence result of the reputational literature.

Regaining an all-in look. In our framework, contestants who give up their all-in look cannot regain it; e.g., an organized criminal group showing weakness once ruins its reputation forever. This assumption captures, in a stylized way, the common wisdom that a beneficial reputation is much harder to gain and sustain than to lose and, conversely, a detrimental reputation is difficult to get rid of (e.g., Levine, 2021). This assumption buys tractability, but in some applications an all-in look can be regained: a new particularly violent member of an organized criminal group may be capable of changing the group's culture and re-establishing the lost reputation. If one would allow contestants to regain an all-in look stochastically over time, then one could extend our results if contestants regaining an all-in look acquired the level of belief in the eyes of others equal to the current belief of the contestants with an all-in look. However, in the perhaps more realistic setup where an all-in look can be regained as a clean slate with belief updating starting once again from the prior ε, then we would incur in the tractability issue already highlighted in the previous paragraph, as contestants may enter a certain period with different levels of (positive) beliefs.

Type-asymmetric equilibria. We focus on equilibria where any two rational contestants entering a period with identical all-in looks behave identically. One-shot all-pay auctions with three or more contestants often have asymmetric equilibria; see Baye, Kovenock, and De Vries, (1996, henceforth, BKD). Thus, we consider the simplest special cases of interest; namely, $T = 2$ and $n = 3$. A complete proof of the following discussion is at the end of the Appendix. When $\varepsilon = 0$, all-in looks play no role and payoffs in $t = 2$ are 0; in $t = 1$, there is a continuum of asymmetric equilibria (without all-in efforts), as in BKD. Similarly, when ε is sufficiently high, the fear of all-in automatons is high and thus all-in efforts are not played even in $t = 1$, so that payoffs in $t = 2$ are 0; in $t = 1$, there is a continuum of asymmetric equilibria (without all-in efforts), once again as in BKD. Interestingly, in the remaining intermediate region of ε, all-in efforts are part of the equilibrium strategies in $t = 1$ and the payoff in $t = 2$ is positive for the monopolist as in Proposition 2; in particular, in $t = 1$, there is a continuum of asymmetric equilibria, where the three contestants exert all-in effort w.p. q and, with the remaining probability $1 - q$, two of them further mix over

$[0, \bar{d}]$, and the third over $[\underline{d}, \bar{d}]$, with $0 < \underline{d} < \bar{d} < 1$, and 0.[28] The equilibrium value of such q is the real root of a polynomial, but cannot be expressed in radicals without complex numbers ("casus irreducibilis"). This suggests that a full equilibrium characterization of type-asymmetric equilibria in the general (T, n, ε)-game is not tractable.

All-in fighting. We assumed that the effort of contestants fighting all-in is 1. This is a natural choice to capture an all-in fight; 1 is the greatest effort in a one-shot all-pay auction that is not strictly dominated by 0 as any effort strictly greater than the prize value would necessarily result in a strictly negative payoff. Consider now the model generalization where the effort exerted by the automatons is $\omega \geq 0$. If $\omega > 1$, fighting all-in would yield a strictly negative per-period payoff, and if $\omega < 1$ it may yield a strictly positive per-period payoff, thus "polluting" the reputational motives to fight all-in; the choice of $\omega = 1$ avoids these two contingencies. Nevertheless, if $\omega > 1$, our equilibrium characterization in Section 5 can be easily extended: as intuition suggests, a higher ω would increase the costs of fighting all-in in early periods and the probability that contestants with an all-in look fight all-in would decrease. Also, in the limit where ω approaches 1 from above all our results carry over. If instead $\omega < 1$, then an equilibrium would not exist, in general, because of profitable deviations slightly above ω.[29]

Discounting. Our analysis can be extended to include a discount factor $\delta < 1$ for all contestants and periods. Intuitively, discounting reduces the benefits of building an all-in look, similarly to a front-loaded prize sequence. For illustration purposes, consider discounting when $T = n = 2$ and first-period reputation $\varepsilon_1 > 0$ for both contestants. As for the second period, nothing changes with respect to the analysis in Section 4.1. As for the first period, it is still true that the upper bound of the support of the uniform distribution and the size of the mass point at 1 are endogenously determined so that the payoff of any effort in the support of the mixed strategy is 0. However, the monopolist's expected second-period payoff is now $\delta\varepsilon_2$, rather than ε_2. Hence, using $q \in [0, 1]$ for the mass at 1 and a for the upper bound of the uniform distribution, we have that the zero-payoff condition: (1) for a bid of a, reads $(1 - \varepsilon)(1 - q) - a = 0$, as bidding a wins with certainty against non-all-in efforts,

[28] Allowing contestants to have different mass points at all-in effort raises the tractability issue highlighted in the two previous extensions: contestants enter a period with different levels of (positive) all-in looks.

[29] To intuitively illustrate why an equilibrium qualitatively similar to that depicted in Figure 2 cannot be sustained, consider a first-period equilibrium behavior that resembles that in panel (a) of Figure 2, except that the mass point is shifted down to $\omega < 1$. Because of competition between the two oligopolists, their cumulative payoffs would be 0. Then, a contestant who deviates to an effort arbitrarily close to ω, but strictly above it, would have a strictly positive per-period profit, contradicting equilibrium. If instead ω was equal to 1, as in our main setup, then such a slightly upward deviation would not be profitable as it would result in a strictly negative per-period payoff.

and (2) for a bid of 1, reads $(\varepsilon + (1-\varepsilon)\,q)\,(1/2+0) + (1-\varepsilon-(1-\varepsilon)\,q)\,(1+\delta\varepsilon_2) - 1 = 0$, where the first addend accounts for first-period ties (yielding no second-period benefits) and the second for the first-period probability that the rival exerts non-all-in effort (yielding second-period discounted monopolistic benefits). Solving (2), using $\varepsilon_2 = \varepsilon/\,(\varepsilon + (1-\varepsilon)\,q)$ by Bayes' rule, we obtain $q = \left(\sqrt{(1+\delta\varepsilon)^2 - 1} - (1+\delta)\,\varepsilon\right)/\,(1-\varepsilon)$, which increases in δ. In words, as opposed to the case of $\delta = 1$, the introduction of discounting decreases the mass at 1 because future reputational benefits are discounted. In line with this result, by (1), the introduction of discounting increases a, as the fear of one's rival all-in effort is reduced.[30]

7 Conclusions

We analyze repeated contests where contestants have the chance of building a tough look through their choice of efforts. The importance of dynamic reputation effects in real life conflicts is well-established. Such importance is also pointed out by Donohue and Levitt (1998; p. 463), whose model of illegal markets "omits a number of potentially important considerations (e.g., private information and dynamic reputation effects)." Those considerations are our main focus. Formally, we analyze the effects of tough looks by equipping a repeated, standard all-pay auction with commitment types à la Kreps and Wilson (1982) and Milgrom and Roberts (1982). Two strengths of our approach emerge. First, using commitment types enhances tractability when compared to models of private information in repeated auctions where signaling is two-directional: weak types may want to appear strong and strong types may want to appear weak.[31] In our model with commitment types, signaling is one-directional: rational types may want to appear as "all-in" fighters. This enhances tractability, allows us to derive rich dynamics, and matches applications where reputation for toughness, rather than for weakness, plays a key role. Second, our use of a standard all-pay auction with continuous fighting efforts allows for a nuanced analysis of efforts, as compared to binary-choice models.

We fully characterize the unique type-symmetric equilibrium of the repeated all-pay auction with commitment types and study its properties in terms of dynamics and efforts. First, contestants actively build an all-in look by fighting hard in a publicly observed manner. The

[30] Note that q is positive if $\varepsilon \leq 1 - 1/\,(1+2\delta)$, which means that an all-in effort is exerted by a rational contestant only if ε is below an upper bound that becomes smaller as discounting is introduced. In other words, the introduction of discounting shrinks the interval in beliefs for which reputation is maintained in a reputational duopoly.

[31] E.g., Hörner and Sahuguet (2007; p. 175) state: "players' incentives to misrepresent their valuations in the first stage are complex, since both sandbagging and bluffing strategies are used in equilibrium." See, also, Kubitz (2022).

importance of building a tough look in real-life repeated contests is further stressed by the fact that, in those applications where actions are not necessarily publicly observed, contestants themselves tend to promote their actions' observability. For instance, disclosure of past litigation in trials is essential; e.g., "PAEs commonly attempt to highlight their willingness to litigate aggressively ... by referencing previous situations in which they have litigated" (Hovenkamp, 2013; p. 3). Second, fighting to build an all-in look is particularly costly: acts of heinous violence in fights among organized criminal groups involve steep fighting costs because they are typically wasteful, risky, and disruptive of commerce; for instance, Levitt and Venkatesh (2000) detail costs and benefits of a drug-selling street gang. Third, such costs of fighting to build an all-in look may outweigh the present expected benefits and yield present losses. This is in line with warring criminal organizations' profits being often negative (Levitt and Venkatesh, 2000).[32] Fourth, such present losses of fighting for all-in looks are offset by the prospect of future benefits; in fact, a contestant who successfully invested in her all-in look can cash in on her all-in look without further exerting costly all-in efforts. An all-in monopolist can cash in on her all-in look and obtain a strictly positive payoff by exerting a non-all-in effort: indeed, a monopolist wins with non-negligible probability by exerting a negligible effort because other contestants concede the fight by exerting effort 0. In other words, an all-in look, once built to high levels, can be used to intimidate rivals and mitigate the need for further acts of heinous costly violence. This finding is in line with evidence from the literature on criminal organizations. For instance, "many mafiosi have begun their careers with violent acts (Hess 1973; Arlacchi 1983), but have subsequently relied on the reputation with which such acts provided them" (Gambetta, 2000).

While, in the applications we discussed, the benefit from having built an all-in look may be long-lasting, in our framework such a benefit is cashed-in in a single period (i.e., when the monopolist's effort is non-all-in). This equilibrium property is the result of our stylized setup where a single period of non-all-in effort immediately wipes out a contestant's reputation for toughness. One could take inspiration from the reputational literature and extend our model, for instance, to either non-persistent types, or an interior probability of fighting all-in for the automatons, or imperfectly observable efforts, or the endogenous selection of the all-in fighting strategy; these extensions would add realism to the model as all-in looks would not necessarily be consumed in a single period and cycles of high efforts may emerge. This is an interesting avenue for future research.

[32] Present losses may also occur in litigation: "[p]atent assertion entities (PAEs) ... frequently initiate infringement lawsuits on which they ostensibly have no chance of turning a profit .... a PAE follows through on its seemingly irrational litigation threats in order to develop a litigious reputation" (Hovenkamp, 2013; p. 2). Also, PAEs may "take cases to judgment rather than settle them even though they are very unlikely to win those cases" (Allison et al., 2010; p. 694).

Appendix

Throughout the Appendix, for brevity, we use the auction terminology "bid" rather than the contest terminology "exert effort." We use the notation p_t for the probability that a contestant with an all-in look bids 1 at period t, q_t for the probability that a contestant without an all-in look bids 0, F_t (G_t) for the CDF employed by a contestant with (without) an all-in look conditional on bidding less than 1 (more than 0), and d_t for the upper bound of the support of F_t and/or G_t. These objects take different values in different equilibrium configurations. We refer to Bayes' rule as $\varepsilon_{t+1} = \varepsilon_t / (\varepsilon_t + (1 - \varepsilon_t) p_t)$.

Proof of Proposition 1. See Baye, Kovenock, and De Vries (1996). ■

Proof of Proposition 2. For $t = T$, we will see that $\pi_T^{\nu} > 0 = \pi_T^{-\nu}$. Consider the following strategies: the contestant with an all-in look bids on $[0, d_T]$ with CDF F_T (and obtains a nonnegative payoff), and the other contestants bid 0 w.p. $q_T \in [0, 1]$ and with CDF G_T on $[0, d_T]$ w.p. $1 - q_T$ (and obtain a zero payoff).

An all-in monopolist bidding $x \in [0, d_T]$ obtains $(q_T + (1 - q_T) G_T(x))^{n-1} - x$, but also she obtains q_T^{n-1} by bidding an arbitrarily small, but strictly positive. This gives,

$$G_T(x) = \frac{\left(q_T^{n-1} + x\right)^{\frac{1}{n-1}} - q_T}{1 - q_T}, \tag{9}$$

so $G_T(0) = 0$ and $G_T(d_T) = 1 \iff d_T = 1 - q_T^{n-1}$. Contestants without an all-in look bidding $x \in [0, d_T]$ obtain $(1 - \varepsilon_T) F_T(x) (q_T + (1 - q_T) G_T(x))^{n-2} - x = 0$, which, by (9), gives

$$F_T(x) = \frac{x}{(1 - \varepsilon_T)\left(q_T^{n-1} + x\right)^{\frac{n-2}{n-1}}}, \tag{10}$$

so $F_T(0) = 0$ and $F_T\left(1 - q_T^{n-1}\right) = 1 \iff q_T^{n-1} = \varepsilon_T$. Hence, $q_T^{n-1} = \varepsilon_T = 1 - d_T$, and thus the monopolist bids on $[0, 1 - \varepsilon_T]$ with CDF F_T from (10) and the other contestants bid 0 w.p. $\varepsilon_T^{\frac{1}{n-1}}$ and on $[0, 1 - \varepsilon_T]$ with CDF G_T from (9) w.p. $1 - \varepsilon_T^{\frac{1}{n-1}}$. The monopolist obtains $\pi_T^{\nu} = \varepsilon_T$. This concludes the characterization of the TSPBE in the statement of the proposition for $t = T$. Throughout, we omit the proofs of uniqueness which are standard (similarly to the method of proof in Baye, Kovenock, and De Vries, 1996), but lengthy.

In the remaining of this proof, we focus on $t \in \{2, ..., T - 1\}$. To calculate the equilibrium in period $t \in \{2, .., T - 1\}$, we need cumulative payoffs in $t + 1$. From Proposition 1, if $\nu_{t+1} = 0$, such payoffs are 0 for all contestants in $t + 1$. If $\nu_{t+1} > 0$, we assume that the cumulative payoff of the monopolist in $t+1$ is strictly positive and that of the others is 0; $\pi_{t+1}^{\nu} > 0 = \pi_{t+1}^{-\nu}$. We verify this assumption at the end of the proof.

As the cumulative payoff of the monopolist π_t^{ν} depends on the belief level ε_t, we adopt the notation $\pi_t^{\nu}(\varepsilon_t)$. To determine equilibrium in t, one needs the cumulative payoff $\pi_{t+1}^{\nu}(\varepsilon_{t+1})$, and ε_{t+1} depends on ε_t and p_t (the probability that the monopolist bids 1 in period t). In t, we denote the resulting ε_{t+1} as function of ε_t and p_t by $\varepsilon_{t+1}(\varepsilon_t, p_t)$. To lighten notation, we indicate these dependencies only when clarity requires it.

Consider the following strategies; the monopolist bids on $[0, d_t]$ with CDF F_t w.p. $1-p_t$ and 1 w.p. p_t, and all other contestants bid 0 w.p. q_t and on $[0, d_t]$ with CDF G_t w.p. $1-q_t$. The monopolist bidding an arbitrarily small, but strictly positive, amount obtains q_t^{n-1}, and bidding $x \in (0, d_t]$ obtains $(q_t + (1-q_t) G_t(x))^{n-1} - x$. Hence,

$$G_t(x) = \frac{\left(q_t^{n-1} + x\right)^{\frac{1}{n-1}} - q_t}{1-q_t}. \tag{11}$$

Thus, $G_t(d_t) = 1$ gives

$$q_t^{n-1} = 1 - d_t. \tag{12}$$

Note that the monopolist bidding an arbitrarily small amount gives up her all-in look and hence obtains zero in the next period (see Proposition 1). The monopolist bidding 1 obtains $1 + \pi_{t+1}^{\nu}(\varepsilon_{t+1}(p_t, \varepsilon_t)) - 1 = q_t^{n-1}$, which gives

$$\pi_{t+1}^{\nu}(\varepsilon_{t+1}(p_t, \varepsilon_t)) = q_t^{n-1}. \tag{13}$$

Contestants without an all-in look obtain $(1-\varepsilon_t)(1-p_t)F_t(x)\left(q_t + (1-q_t) G_t(x)\right)^{n-2} - x = 0$ when bidding $x \in [0, d_t]$. Therefore, by (11), we have

$$F_t(x) = \frac{x}{(1-\varepsilon_t)(1-p_t)(q_t^{n-1} + x)^{\frac{n-2}{n-1}}}, \tag{14}$$

and, by (12), $F_t(d_t) = 1$ gives $d_t = (1-\varepsilon_t)(1-p_t)$, which, together with (12) and (13), implies

$$\pi_{t+1}^{\nu}(\varepsilon_{t+1}(p_t, \varepsilon_t)) = q_t^{n-1} = 1 - d_t = \varepsilon_t + (1-\varepsilon_t) p_t. \tag{15}$$

To conclude the equilibrium characterization, we compute cumulative payoffs recursively proving

$$\pi_{t+1}^{\nu}(\varepsilon_{t+1}) = \varepsilon_{t+1}^{\frac{1}{T-t}} \Longrightarrow \pi_t^{\nu}(\varepsilon_t) = \varepsilon_t^{\frac{1}{T-t+1}}. \tag{16}$$

To obtain $\pi_t^\nu(\varepsilon_t)$ from $\pi_{t+1}^\nu(\varepsilon_{t+1}(p_t,\varepsilon_t))$, we use Bayes' rule

$$\varepsilon_{t+1}(p_t,\varepsilon_t) = \frac{\varepsilon_t}{\varepsilon_t + (1-\varepsilon_t)\, p_t}. \tag{17}$$

Next, $\pi_{t+1}^\nu(\varepsilon_{t+1}(p_t,\varepsilon_t)) = (\varepsilon_{t+1}(p_t,\varepsilon_t))^{\frac{1}{T-t}} = \left(\frac{\varepsilon_t}{\varepsilon_t+(1-\varepsilon_t)p_t}\right)^{\frac{1}{T-t}}$ by the hypothesis of (16) and (17). Hence, using the extremes of (15), we have that

$$\begin{aligned} \left(\frac{\varepsilon_t}{\varepsilon_t + (1-\varepsilon_t)\, p_t}\right)^{\frac{1}{T-t}} &= \varepsilon_t + (1-\varepsilon_t)\, p_t, \\ &\Longleftrightarrow \varepsilon_t = (\varepsilon_t + (1-\varepsilon_t)\, p_t)^{T-t+1}, \\ &\Longleftrightarrow \varepsilon_t^{\frac{1}{T-t+1}} = \varepsilon_t + (1-\varepsilon_t)\, p_t. \end{aligned} \tag{18}$$

Hence,

$$\varepsilon_t + (1-\varepsilon_t)\, p_t = \varepsilon_t^{\frac{1}{T-t+1}} = \left(\frac{\varepsilon_t}{\varepsilon_t + (1-\varepsilon_t)\, p_t}\right)^{\frac{1}{T-t}},$$

and $\varepsilon_t + (1-\varepsilon_t)\, p_t = q_t^{n-1}$ by (15). Also,

$$\pi_t^\nu(\varepsilon_t) = q_t^{n-1}, \tag{19}$$

because q_t^{n-1} is the payoff that the monopolist obtains in period t by bidding an arbitrarily small, but strictly positive, amount. Thus, we have

$$\pi_t^\nu(\varepsilon_t) = \varepsilon_t^{\frac{1}{T-t+1}}. \tag{20}$$

The above proves (16). To fully characterize payoffs, we proceed by backward induction and repeatedly use (16). From the above analysis of the last period, $\pi_T^\nu(\varepsilon_T) = \varepsilon_T$. This fits the hypothesis of (16). Thus, $\pi_{T-1}^\nu(\varepsilon_{T-1}) = \varepsilon_{T-1}^{\frac{1}{T-(T-1)+1}} = \sqrt{\varepsilon_{T-1}}$. All previous periods' cumulative payoffs follow similarly. Also, from the above analysis of the last period, $\pi_T^{-\nu}(\varepsilon_T) = 0$. Then, by the considered strategies, in all previous periods $\pi_t^{-\nu}(\varepsilon_t) = 0$. Those facts verify our initial assumption that, if $\nu_{t+1} = 1$, the cumulative payoff of the monopolist in $t+1$ is strictly positive and that of the others is 0.

Having now the full payoff characterization, we complete the strategy characterization. First, (19) and (20) give $q_t = \varepsilon_t^{\frac{1}{(T-t+1)(n-1)}}$. Second, from (12), $d_t = 1 - q_t^{n-1} = 1 - \varepsilon_t^{\frac{1}{T-t+1}}$. Third, from (15),

$$p_t = 1 - \frac{d_t}{1-\varepsilon_t} = \frac{\varepsilon_t^{\frac{1}{T-t+1}} - \varepsilon_t}{1-\varepsilon_t}. \tag{21}$$

Finally, F_t follows from (14) and G_t from (11), so that the strategies match those in the statement of the proposition for $t \in \{2, \ldots, T-1\}$. ■

Proof of Lemma 1. Existence and uniqueness of $\bar{\varepsilon}_t(\nu_t)$ follows immediately from (5): the RHS strictly decreases in ε_t, takes value 2 when $\varepsilon_t = 0$ and value 0 when $\varepsilon_t = 1$. Now, we prove that $\bar{\varepsilon}_t(\nu_t) \leq 2/3$. The RHS of condition (5), evaluated at $\varepsilon_t = 2/3$, is smaller than 1 if and only if

$$\begin{aligned}
\frac{1}{\nu_t}\sum_{l=0}^{\nu_t-1}\left(\frac{1}{3}\right)^l + \left(\frac{1}{3}\right)^{\nu_t-1} &< 1 \\
&\iff \frac{1}{\nu_t}\left(\frac{1-\left(\frac{1}{3}\right)^{\nu_t}}{1-\frac{1}{3}}\right) + \left(\frac{1}{3}\right)^{\nu_t-1} < 1 \\
&\iff \frac{3}{2\nu_t}\left((2\nu_t-1)\left(\frac{1}{3}\right)^{\nu_t}+1\right) \leq 1 \\
&\iff (2\nu_t-1)\left(\frac{1}{3}\right)^{\nu_t-1} \leq 2\nu_t - 3 \\
&\iff \left(\frac{1}{3}\right)^{\nu_t-1} \leq 1 - \frac{2}{2\nu_t-1}.
\end{aligned}$$

The LHS of the above-displayed inequality decreases in ν_t, its RHS increases in ν_t, and the inequality holds at $\nu_t = 2$. Therefore, the RHS of (5) is smaller than 1 for any $\nu_t \geq 2$ and any $\varepsilon_t \geq 2/3$. Hence, the solution of (5) must satisfy $\bar{\varepsilon}_t(\nu_t) \leq 2/3$.

Finally, we prove that $\bar{\varepsilon}_t(\nu_t)$ decreases in ν_t. We can rewrite (5) as

$$\begin{aligned}
1 &= \frac{1-(1-\varepsilon_t)^{\nu_t}}{\varepsilon_t\nu_t} + (1-\varepsilon_t)^{\nu_t-1} \\
&\iff \nu_t\left(1-(1-\varepsilon_t)^{\nu_t-1}\right) = \frac{1}{\varepsilon_t} - \frac{(1-\varepsilon_t)^{\nu_t}}{\varepsilon_t} \\
&\iff \nu_t\left(1-(1-\varepsilon_t)^{\nu_t-1}\right) = \frac{1-\varepsilon_t}{\varepsilon_t} + 1 - \frac{(1-\varepsilon_t)^{\nu_t}}{\varepsilon_t} \\
&\iff \nu_t\left(1-(1-\varepsilon_t)^{\nu_t-1}\right) = \frac{1-\varepsilon_t}{\varepsilon_t}\left(1-(1-\varepsilon_t)^{\nu_t-1}\right) + 1 \\
&\iff \left(\nu_t - \frac{1-\varepsilon_t}{\varepsilon_t}\right)\left(1-(1-\varepsilon_t)^{\nu_t-1}\right) = 1.
\end{aligned}$$

As the above-displayed equation is solved by $\bar{\varepsilon}_t(\nu_t)$, its LHS must be strictly positive at $\bar{\varepsilon}_t(\nu_t)$, so that $\nu_t > (1-\bar{\varepsilon}_t(\nu_t))/\bar{\varepsilon}_t(\nu_t)$. Therefore, around the solution $\varepsilon_t = \bar{\varepsilon}_t(\nu_t)$, the LHS of the above-displayed equation strictly increases in ν_t and in ε_t, and thus $\bar{\varepsilon}_t(\nu_t)$ strictly decreases in ν_t. ■

Proof of Proposition 3. For $t = T$, consider the following strategies; the ν_T contestants with an all-in look bid on $[0, d_T]$ with CDF F_T and the other contestants bid 0. A contestant with an all-in look bidding $x \in [0, d_T]$ obtains $((1 - \varepsilon_T) F_T(x))^{\nu_T - 1} - x = 0$, which gives

$$F_T(x) = \frac{x^{\frac{1}{\nu_T - 1}}}{1 - \varepsilon_T}, \tag{22}$$

and $F_T(d_T) = 1$ yields $d_T = (1 - \varepsilon_T)^{\nu_T - 1}$. Therefore, F_T in (22) matches the statement of the proposition for $t = T$. Any one of the $n - \nu_T$ contestants without an all-in look bidding $x \in (0, d_T]$ obtains $((1 - \varepsilon_T) F_T(x))^{\nu_T} - x < 0 \iff x^{\nu_T/(\nu_T - 1)} - x < 0$; thus, such deviation is not profitable. This concludes the characterization of the TSPBE in period T. Throughout, we omit the proofs of uniqueness which are standard (similarly to the method of proof in Baye, Kovenock, and De Vries, 1996), but lengthy.

In the remaining of this proof, focus on $t \in \{1, ..., T - 1\}$. To calculate equilibrium in $t \in \{1, .., T - 1\}$, we need cumulative payoffs in $t + 1$. From Proposition 1, if $\nu_{t+1} = 0$, then cumulative payoffs in $t + 1$ are 0 for all contestants. As shown in Proposition 2, if $\nu_t = 1$, then $\pi_t^{\nu} > 0 = \pi_t^{-\nu}$. Assume that, in any t, if $\nu_t \geq 2$, then all cumulative payoffs are 0. We verify this assumption at the end of the proof.

As the cumulative payoff of a contestant with an all-in look π_t^{ν} depends on the belief level ε_t, we adopt notation $\pi_t^{\nu}(\varepsilon_t)$. To determine equilibrium in t, one needs the cumulative payoff $\pi_{t+1}^{\nu}(\varepsilon_{t+1})$, and ε_{t+1} depends on ε_t and p_t (the probability that the monopolist bids 1 in period t). In t, we denote the resulting ε_{t+1} as function of ε_t and p_t by $\varepsilon_{t+1}(\varepsilon_t, p_t)$. To lighten notation, we indicate these dependencies only when clarity requires it.

Consider the following strategies: contestants with an all-in look bid with CDF F_t on $[0, d_t]$ w.p. $1 - p_t$ and 1 w.p. p_t, and contestants without an all-in look bid 0. A contestant with an all-in look bidding $x \in [0, d_t]$ obtains $((1 - \varepsilon_t)(1 - p_t) F_t(x))^{\nu_t - 1} - x = 0$, implying

$$F_t(x) = \frac{x^{\frac{1}{\nu_t - 1}}}{(1 - \varepsilon_t)(1 - p_t)}. \tag{23}$$

Note that the contestant with an all-in look bidding an arbitrarily small amount gives up her all-in look and hence obtains zero in the next period (see Proposition 1).

A contestant without an all-in look bidding $x \in [0, d_t]$ obtains $(1 - \varepsilon_t)^{\nu_t} (1 - p_t)^{\nu_t} F_t(x)^{\nu_t} - x < 0$, which holds true by (23), and thus she has no incentive to deviate to bidding $x \in [0, d_t]$. Since $F_t(d_t) = 1$, we have that, using (23),

$$d_t = ((1 - \varepsilon_t)(1 - p_t))^{\nu_t - 1}. \tag{24}$$

(23) and (24) give the strategy in the statement of the proposition for $t \in \{1, ..., T-1\}$.

Next, we determine p_t. A contestant with an all-in look bidding 1 must obtain a cumulative payoff equal to 0; this implies that the cost of 1 must equal the future cumulative payoff in case of becoming a monopolist plus the current period's prize accounting for ties, or

$$\begin{aligned}
&1 = (1-\varepsilon_t)^{\nu_t - 1} (1-p_t)^{\nu_t - 1} \pi_{t+1}^{\nu} (\varepsilon_{t+1}(p_t, \varepsilon_t)) \\
&\quad + \sum_{l=0}^{\nu_t - 1} \frac{1}{l+1} \begin{pmatrix} \nu_t - 1 \\ l \end{pmatrix} (\varepsilon_t + (1-\varepsilon_t) p_t)^l (1 - (\varepsilon_t + (1-\varepsilon_t) p_t))^{\nu_t - 1 - l} \\
&\iff 1 = \frac{1}{\nu_t} \sum_{l=0}^{\nu_t - 1} (1-\varepsilon_t)^l (1-p_t)^l + (1-\varepsilon_t)^{\nu_t - 1} (1-p_t)^{\nu_t - 1} \pi_{t+1}^{\nu}(\varepsilon_{t+1}(p_t, \varepsilon_t)),
\end{aligned}$$

which is equivalent to (4) as $\pi_{t+1}^{\nu}(\varepsilon_{t+1}) = \varepsilon_{t+1}^{\frac{1}{T-t}} = \left(\frac{\varepsilon_t}{\varepsilon_t + (1-\varepsilon_t) p_t}\right)^{\frac{1}{T-t}}$ (by Proposition 2 and Bayes' rule). Thus, the above-displayed equation can be written as

$$\frac{1 - \frac{1 - (1 - (\varepsilon_t + p_t(1-\varepsilon_t)))^{\nu_t}}{(\varepsilon_t + p_t(1-\varepsilon_t))\nu_t}}{(1 - (\varepsilon_t + p_t(1-\varepsilon_t)))^{\nu_t - 1}} = \left(\frac{\varepsilon_t}{\varepsilon_t + (1-\varepsilon_t) p_t}\right)^{\frac{1}{T-t}}. \tag{25}$$

Using notation $z \equiv 1 - (\varepsilon_t + p_t(1-\varepsilon_t))$, we rewrite (25) as

$$\begin{aligned}
1 - \frac{1 - z^{\nu_t}}{(1-z)\nu_t} &= z^{\nu_t - 1} \left(\frac{\varepsilon_t}{1-z}\right)^{\frac{1}{T-t}}, \\
1 - \frac{1 + z + z^2 + ... + z^{\nu_t - 1}}{\nu_t} &= z^{\nu_t - 1} \left(\frac{\varepsilon_t}{1-z}\right)^{\frac{1}{T-t}}.
\end{aligned} \tag{26}$$

We next show that a solution $z \in [0, 1-\varepsilon_t]$ to (26) exists if and only if ε_t is sufficiently small, and this solution is unique. It then follows that, when a solution $z \in [0, 1-\varepsilon_t]$ to (26) exists, then there is a unique solution $p_t \in [0,1]$ to (25).

The LHS of (26) strictly decreases in z and the RHS strictly increases. Thus, at most one solution exists. If $z = 0$ (or $p_t = 1$), the LHS of (26) is strictly greater than its RHS. Then, a unique solution $z \in [0, 1-\varepsilon_t]$ to (26) exists if and only if, at $z = 1-\varepsilon_t$ (or $p_t = 0$), the LHS of (26) is strictly smaller than its RHS, or equivalently,

$$\begin{aligned}
&1 - \frac{1 + (1-\varepsilon_t) + ... + (1-\varepsilon_t)^{\nu_t - 1}}{\nu_t} < (1-\varepsilon_t)^{\nu_t - 1} \\
&\iff \nu_t \left(1 - (1-\varepsilon_t)^{\nu_t - 1}\right) < 1 + (1-\varepsilon_t) + ... + (1-\varepsilon_t)^{\nu_t - 1} \\
&\iff \nu_t \left(1 - (1-\varepsilon_t)^{\nu_t - 1}\right) < \frac{1 - (1-\varepsilon_t)^{\nu_t}}{\varepsilon_t}.
\end{aligned} \tag{27}$$

We now show that (27) is satisfied if and only if ε_t is sufficiently small (later, we consider the remaining case of ε_t sufficiently large); that is, $\varepsilon_t \leq \bar{\varepsilon}_t(\nu_t)$, where $\bar{\varepsilon}_t(\nu_t)$ is the unique solution for ε_t of (4) with $p_t = 0$, or

$$\nu_t \left(1 - (1-\varepsilon_t)^{\nu_t - 1}\right) = \frac{1 - (1-\varepsilon_t)^{\nu_t}}{\varepsilon_t},$$

which is equivalent to (5) by $\frac{1-(1-\varepsilon_t)^{\nu_t}}{\varepsilon_t} = \sum_{l=0}^{\nu_t - 1} (1-\varepsilon_t)^l$. The above-displayed equation is uniquely solved by $\varepsilon_t = \bar{\varepsilon}_t(\nu_t)$ by Lemma 1. Now, note that the LHS of the above-displayed equation increases in ε_t and its RHS decreases as it equals $\sum_{l=0}^{\nu_t-1}(1-\varepsilon_t)^l$. Therefore, (27) is satisfied for $\varepsilon_t \leq \bar{\varepsilon}_t(\nu_t)$. Hence, we proved that, $\forall \varepsilon_t \leq \bar{\varepsilon}_t(\nu_t)$, the unique equilibrium is as in the statement of the proposition. To conclude the equilibrium characterization for $\varepsilon_t \leq \bar{\varepsilon}_t(\nu_t)$, note that p_t is characterized by (25), and then p_t and (24) give d_t.

We now consider ε_t large; that is, $\varepsilon_t > \bar{\varepsilon}_t(\nu_t)$. Proceeding as for the case of $t = T$, one can show that, if all other rational contestants use F_t with $p_t = 0$ as described in the proposition, then any bid $x \in \left[0, (1-\varepsilon_t)^{\nu_t - 1}\right]$ yields 0. A deviation to an all-in effort 1, yields

$$\frac{1-(1-\varepsilon_t)^{\nu_t}}{\varepsilon_t \nu_t} + (1-\varepsilon_t)^{\nu_t - 1} - 1. \tag{28}$$

Note that, as discussed after (27) above, $\varepsilon_t > \bar{\varepsilon}_t(\nu_t)$ implies

$$\nu_t \left(1 - (1-\varepsilon_t)^{\nu_t - 1}\right) > \frac{1 - (1-\varepsilon_t)^{\nu_t}}{\varepsilon_t},$$

which in turn implies that (28) is negative. Thus, a deviation to 1 is not profitable. Therefore, when $\varepsilon_t > \bar{\varepsilon}_t(\nu_t)$, the equilibrium strategies are as in the $t = T$ case.

Next, we prove that $p_t \in [0, 2/3]$. Equation (25) is equivalent to

$$\frac{(\varepsilon_t + p_t(1-\varepsilon_t))(1-(\varepsilon_t + p_t(1-\varepsilon_t)))^{\nu_t - 1}}{(\varepsilon_t + p_t(1-\varepsilon_t))\nu_t - 1 + (1-(\varepsilon_t + p_t(1-\varepsilon_t)))^{\nu_t}} = \frac{\varepsilon_t \left(\frac{\varepsilon_t + (1-\varepsilon_t)p_t}{\varepsilon_t}\right)^{\frac{T-t+1}{T-t}}}{\nu_t(\varepsilon_t + (1-\varepsilon_t)p_t)}. \tag{29}$$

We now show that the LHS of (29) decreases in p_t and the RHS increases. The latter follows because $(T-t+1)/(T-t) > 1$. To see that the LHS of (29) decreases in p_t for fixed ε_t, use the notation $z = 1 - (\varepsilon_t + p_t(1-\varepsilon_t))$ to write the LHS of (29) as

$$\frac{(1-z)\, z^{\nu_t - 1}}{(1-z)\,\nu_t - 1 + z^{\nu_t}}. \tag{30}$$

The derivative of (30) with respect to z has the same sign of $\kappa(\nu_t)$ with

$$\kappa(\nu_t) \equiv -2\nu_t - z^{\nu_t} + \nu_t(\nu_t + z(-2\nu_t + (\nu_t - 1)z + 3)) + 1,$$

and we now show that $\kappa(2) > 0$ and $\kappa(\nu_t + 1) > \kappa(\nu_t)$, concluding the proof that the LHS of (29) decreases in p_t. Indeed, $\kappa(2) = (1-z)^2 > 0$ and $\kappa(\nu_t + 1) > \kappa(\nu_t) \iff (1-z)(2\nu_t(1-z) - (1 - z^{\nu_t})) > 0 \iff 2\nu_t - (1 + z + ... + z^{\nu_t - 1}) > 0$, which holds true because $1 + z + ... + z^{\nu_t-1} \leq \nu_t$ by $z < 1$. This shows that the LHS of (29) decreases in p_t and the RHS increases.

Next, we know that, if $p_t = 0$, the LHS of (29) is larger than the RHS of (29) if and only if $\varepsilon_t\nu_t\left(1 - (1-\varepsilon_t)^{\nu_t - 1}\right) < (1 - (1-\varepsilon_t)^{\nu_t})$ which holds true (see the discussion after (26)). We now show that there is no possible solution of (29) with $p_t > 2/3$ by showing that, if $p_t = 2/3$, the LHS of (29) is still smaller than the RHS. As the RHS of (29) increases in $T-t$, it is greater than its value at $T-t = 1$, which is $1/\varepsilon_t(1/3 + 2/(3\varepsilon_t))$. This value is in turn greater than the LHS of (29) if and only if $9\varepsilon_t\nu_t(1-\varepsilon_t)^{\nu_t-1} < 3(1-\varepsilon_t)^{\nu_t} + 3^{\nu_t}((\varepsilon_t + 2)\nu_t - 3)$, which holds true because its LHS is maximized at $\varepsilon_t = 1/\nu_t$, where it takes value $9\left((\nu_t - 1)/\nu_t\right)^{\nu_t - 1} < 9$, and its RHS is larger than 9 as $3(1-\varepsilon_t)^{\nu_t} + 3^{\nu_t}((\varepsilon_t + 2)\nu_t - 3) > 3^{\nu_t}((0+2)\nu_t - 3) \geq 3^2 = 9$. Hence, we proved that for any ε_t, $p_t \leq 2/3$.

Finally, we assumed that, if $\nu_t \geq 2$, $\pi_t^\nu = 0$. It remains to show that there is no equilibrium such that $\pi_t^\nu = \rho > 0$; if so, the analogue of (23) would be

$$\pi_t^\nu = ((1-\varepsilon_t)(1-p_t)F_t(x))^{\nu_t - 1} - x = \rho \iff F_t(x) = \frac{x^{\frac{1}{\nu_t - 1}} + \rho}{(1-\varepsilon_t)(1-p_t)},$$

and hence contestants with an all-in look would bid 0 with strictly positive probability, which is a contradiction because contestants with an all-in look would then find it profitable to bid an arbitrarily small, but strictly positive, amount. ■

Proof of Proposition 4. We start with the second statement. By Bayes' rule,

$$\varepsilon_{t+1} = \frac{\varepsilon_t}{\varepsilon_t + (1-\varepsilon_t)p_t}.$$

By Proposition 2,

$$p_t = \frac{\varepsilon_t^{\frac{1}{T-t+1}} - \varepsilon_t}{1 - \varepsilon_t}.$$

Hence, $\varepsilon_{t+1} = \varepsilon_t^{\frac{T-t}{T-t+1}}$. Now, since $\frac{T-t}{T-t+1} \leq 1$ and $\varepsilon_t \in (0,1)$, $\varepsilon_{t+1} > \varepsilon_t$. The first statement follows as $\varepsilon_t > 0$ implies $p_t > 0$ in every non-terminal period of all-in monopoly. ■

Proof of Proposition 5. We use the equilibrium described in Proposition 2 and notation $L \equiv T - t + 1$ to denote the number of periods remaining from t onwards. We obtain that the statement of the proposition is equivalent to

$$\begin{aligned}
& \tfrac{1-\varepsilon_t^{\frac{1}{L}}}{1-\varepsilon_t} \int_0^{1-\varepsilon_t^{\frac{1}{L}}} x dF_t(x) + \tfrac{\varepsilon_t^{\frac{1}{L}}-\varepsilon_t}{1-\varepsilon_t} + (n-1)\left(1-\varepsilon_t^{\frac{1}{L(n-1)}}\right) \int_0^{1-\varepsilon_t^{\frac{1}{L}}} x dG_t(x) \leq 1 \\
\iff \quad & \tfrac{1}{1-\varepsilon_t} \int_0^{1-\varepsilon_t^{\frac{1}{L}}} x d\left(\frac{x}{\left(\varepsilon_t^{\frac{1}{L}}+x\right)^{\frac{n-2}{n-1}}}\right) + \tfrac{\varepsilon_t^{\frac{1}{L}}-\varepsilon_t}{1-\varepsilon_t} + (n-1) \int_0^{1-\varepsilon_t^{\frac{1}{L}}} x d\left(\varepsilon_t^{\frac{1}{L}}+x\right)^{\frac{1}{n-1}} \leq 1 \\
\iff \quad & \tfrac{1}{1-\varepsilon_t}\left(-\varepsilon_t - 2\varepsilon_t^{\frac{1}{L}} + \varepsilon_t^{\frac{2}{L}} + n\varepsilon_t^{\frac{1}{L}} - \tfrac{(n-1)^2}{n}\left(\varepsilon_t^{\frac{1}{L}}\right)^{\frac{n}{n-1}} + \tfrac{1}{n}\right) \leq 1 \\
\iff \quad & (n-1)\left(n\varepsilon_t^{\frac{1}{L}} - (n-1)\left(\varepsilon_t^{\frac{1}{L}}\right)^{\frac{n}{n-1}} - 1\right) \leq n\left(\varepsilon_t^{\frac{1}{L}} - \varepsilon_t^{\frac{2}{L}}\right).
\end{aligned}$$

The RHS of the above-displayed equation is positive and its LHS is negative because it takes value 0 as $\varepsilon_t = 1$ and

$$\frac{d\left(n\varepsilon_t^{\frac{1}{L}} - (n-1)\left(\varepsilon_t^{\frac{1}{L}}\right)^{\frac{n}{n-1}} - 1\right)}{d\varepsilon_t^{\frac{1}{L}}} = n\left(1 - \left(\varepsilon_t^{\frac{1}{L}}\right)^{\frac{1}{n-1}}\right) > 0.$$

This concludes the proof. ■

Proof of Proposition 6. The unique equilibrium is characterized in Proposition 2. Let $\tilde{G}_t(x) \equiv q_t + (1-q_t) G_t(x)$. We obtain

$$\tilde{G}_t(x) = \left(\varepsilon_t^{\frac{1}{T-t+1}} + x\right)^{\frac{1}{n-1}}.$$

Recalling that $L = T - t + 1$, the monopolist wins in period t with probability

$$\begin{aligned}
& \frac{\varepsilon_t^{\frac{1}{L}} - \varepsilon_t}{1-\varepsilon_t} + \frac{1-\varepsilon_t^{\frac{1}{L}}}{1-\varepsilon_t} \int_0^{1-\varepsilon_t^{\frac{1}{L}}} \tilde{G}_t(x)^{n-1} dF_t(x) \\
= \quad & \frac{1}{n} + \frac{\varepsilon^{\frac{1}{L}} - (n-1)^2 \varepsilon^{\frac{n}{L(n-1)}} + (n-1)n\varepsilon^{\frac{1}{L}} - n\varepsilon^{\frac{1}{L}}}{n\left(1-\varepsilon^{\frac{1}{L}}\right)} \\
= \quad & \frac{1}{n} + (n-1)\frac{-(n-1)\varepsilon^{\frac{n}{L(n-1)}} + n\varepsilon^{\frac{1}{L}} - \varepsilon^{\frac{1}{L}}}{n\left(1-\varepsilon^{\frac{1}{L}}\right)},
\end{aligned}$$

which is equal to the expression in (6). ■

Proof of Corollary 1. Letting $\phi \equiv \varepsilon_t^{\frac{1}{T-t+1}}$, (6) is equivalent to

$$\frac{1}{n} + \frac{(n-1)^2}{n} \frac{\phi - \phi^{\frac{n}{n-1}}}{1-\phi}, \tag{31}$$

which is increasing in ϕ as its derivative with respect to ϕ is

$$\frac{(n-1)\left((\phi - n)\phi^{\frac{1}{n-1}} + (n-1)\right)}{n(\phi-1)^2},$$

which, in turn, is positive because its numerator is decreasing in ϕ and takes value 0 at $\phi = 1$. Note that ϕ increases in ε_t and decreases in t (for fixed ε_t). Hence, the derivative of (31) with respect to n, which we label $\beta(\phi) \equiv -\phi^{\frac{n}{n-1}}(n^2 - n\log(\phi) - 1) + n^2\phi - 1$, is negative as $\lim_{\phi\to 0} \beta(\phi) = -1 \leq 0$, $\beta(1) = 0$, and $\partial^2\beta(\phi)/\partial\phi^2 = \frac{n^2\phi^{\frac{1}{n-1}-1}\log(\phi)}{(n-1)^2} < 0$. ■

Proof of Proposition 7. We begin with the proof of part 1. of the proposition. When $\varepsilon_t \in [\bar{\varepsilon}_t(\nu_t), 1]$, from Proposition 3, in $\tau = t+1$ no contestant has an all-in look and the result follows. Hence, we focus on $\varepsilon_t \in (0, \bar{\varepsilon}_t(\nu_t)]$, a range where contestants may bid 1 in equilibrium. Recall from Lemma 1 that $\bar{\varepsilon}_t(\nu_t) \leq 2/3$, and hence $\nu_\tau \geq 2$ is not sustainable whenever $\varepsilon_\tau > 2/3$. Thus, we derive the maximum number of periods to sustain $\nu_\tau \geq 2$ by focusing on the slowest possible path of increase in ε_τ. As $p_t \leq 2/3$ from Proposition 3, Bayes' rule implies that the slowest possible increase of ε between periods t and $t+1$ is given by

$$\varepsilon_{t+1} \geq \frac{\varepsilon_t}{\varepsilon_t + (1-\varepsilon_t)(2/3)} = \frac{3\varepsilon_t}{\varepsilon_t + 2}.$$

Therefore, $\forall \tau \geq t+1$, the lowest possible value of ε_τ equals

$$\varepsilon_\tau = \frac{3^{\tau-t}\varepsilon_t}{2^{\tau-t}(1-\varepsilon_t) + 3^{\tau-t}\varepsilon_t}, \tag{32}$$

as the above solves the difference equation $\varepsilon_{n+1} = \frac{3\varepsilon_n}{\varepsilon_n+2}$ initialized at $\varepsilon_n = \varepsilon_t$. Thus, even an increase of ε from period t to τ along the slowest possible path surpasses 2/3 when $\varepsilon_\tau \geq 2/3$ in (32), or

$$\begin{aligned} 2\left(2^{\tau-t}(1-\varepsilon_t) + 3^{\tau-t}\varepsilon_t\right) \quad &\geq \quad 3^{\tau-t+1}\varepsilon_t \iff 2^{\tau-t+1}(1-\varepsilon_t) + 2 * 3^{\tau-t}\varepsilon_t \geq 3^{\tau-t+1}\varepsilon_t \\ &\iff \quad 2^{\tau-t+1}(1-\varepsilon_t) \geq 3^{\tau-t}\varepsilon_t, \end{aligned}$$

which is equivalent to (7). This concludes the proof of part 1. of the proposition.

To prove part 2. of the proposition, note that (4) can be written as

$$1 = \frac{1-(1-(\varepsilon_t + p_t(1-\varepsilon_t)))^{\nu_t}}{(\varepsilon_t + p_t(1-\varepsilon_t))\nu_t} + (1-(\varepsilon_t + p_t(1-\varepsilon_t)))^{\nu_t - 1}\left(\frac{\varepsilon_t}{\varepsilon_t + p_t\,(1-\varepsilon_t)}\right)^{\frac{1}{T-t}}. \quad (33)$$

We now write Bayes' rule as

$$\varepsilon_t + (1-\varepsilon_t)\,p_t = 1-(1-\varepsilon_t)\,(1-p_t) = \frac{\varepsilon_t}{\varepsilon_{t+1}},$$

and use this to rewrite (33) as (8). Note that existence and uniqueness of ε_{t+1} in (8) follow from existence and uniqueness of p_t demonstrated in Proposition 3 and Bayes' rule. Furthermore, Bayes' rule, together with $p_t \leq 2/3$ (see Proposition 3), implies that $\varepsilon_{t+1} > \varepsilon_t$. ■

Proof of Proposition 8. We use the equilibrium described in Proposition 3, where we focus on $\varepsilon_t \in (0, \bar{\varepsilon}_t\,(\nu_t)]$. The strategy of proof is to show that aggregate effort converges to 1 when $\varepsilon_t \to 0$ and increases in ε_t when $\varepsilon_t = 0$. First, we calculate the limit of p_t as $\varepsilon_t \to 0$. Second, we derive the key quantity of interest $\lim_{\varepsilon_t \to 0}\,(1-z)\,\frac{dz}{d\varepsilon_t}$, which helps us to characterize the behavior of aggregate effort near $\varepsilon_t = 0$. Third, we use these two to characterize aggregate effort.

First, using (26) in the Proof of Proposition 3 and notation $z \equiv 1-(\varepsilon_t + (1-\varepsilon_t)\,p_t)$, if $\varepsilon_t \to 0$, then the only solution is $z \to 1$, which implies $p_t \to 0$.

Second, take limits as $\varepsilon_t \to 0$ in (26) and obtain

$$\lim_{\varepsilon_t \to 0}\left(1 - \frac{1+z+z^2+...+z^{\nu_t - 1}}{\nu_t}\right) = 1 - \frac{\nu_t}{\nu_t} = 0 = \lim_{\varepsilon_t \to 0} z^{\nu_t - 1}\left(\frac{\varepsilon_t}{1-z}\right)^{\frac{1}{T-t}},$$

and therefore $\lim_{\varepsilon_t \to 0}\,(\varepsilon_t/\,(1-z))^{\frac{1}{T-t}} = 0$. Now, rewrite (26) as

$$\frac{\left(\frac{1-\frac{1+z+z^2+...+z^{\nu_t-1}}{\nu_t}}{z^{\nu_t-1}}\right)^{T-t}}{1-z} = \frac{\varepsilon_t}{(1-z)^2}. \quad (34)$$

Taking limits again and using L'Hôpital's rule, the LHS of (34), recalling that $z \to 1$ when

$\varepsilon_t \to 0$, equals

$$\lim_{z\to 1} \frac{(T-t)\left(\frac{1-\frac{1+z+z^2+...+z^{\nu_t-1}}{\nu_t}}{z^{\nu_t-1}}\right)^{T-t-1} \frac{-\frac{1}{\nu_t}\sum_{l=1}^{\nu_t-1} l z^{l-1} z^{\nu_t-1} - (\nu_t-1)\left(1-\frac{1+z+z^2+...+z^{\nu_t-1}}{\nu_t}\right) z^{\nu_t-2}}{\left(z^{\nu_t-1}\right)^2}}{-1}$$
$$= \begin{cases} \frac{\nu_t-1}{2} & \text{if } T-t=1, \\ 0 & \text{if } T-t \geq 2. \end{cases}$$

Similarly, by L'Hôpital's rule, the RHS of (34) is equivalent to

$$\lim_{\varepsilon_t \to 0} \frac{\varepsilon_t}{(1-z)^2} = \lim_{\varepsilon_t \to 0} \frac{1}{-2(1-z)\frac{dz}{d\varepsilon_t}}.$$

Hence, if $T-t=1$,

$$\lim_{\varepsilon_t \to 0} (1-z)\frac{dz}{d\varepsilon_t} = \frac{1}{1-\nu_t}, \tag{35}$$

and if $T-t \geq 2$,

$$\lim_{\varepsilon_t \to 0} (1-z)\frac{dz}{d\varepsilon_t} = -\infty. \tag{36}$$

Third, recall from Proposition 3 that contestants without an all-in look bid 0. Denoting as $X_t(\varepsilon_t)$ the expected bid in period t of a contestant with an all-in look, we obtain

$$\begin{aligned} X_t(\varepsilon_t) \quad &= \quad (1-p_t)\int_0^{((1-\varepsilon_t)(1-p_t))^{\nu_t-1}} x * d\frac{x^{\frac{1}{\nu_t-1}}}{(1-\varepsilon_t)(1-p_t)} + p_t \\ &\Longleftrightarrow \quad X_t(\varepsilon_t) = \frac{(1-\varepsilon_t)^{\nu_t-1}(1-p_t)^{\nu_t}}{\nu_t} + p_t \\ &\Longleftrightarrow \quad \nu_t X_t(\varepsilon_t) - \nu_t = \frac{z^{\nu_t}}{1-\varepsilon_t} - \nu_t(1-p_t) \\ &\Longleftrightarrow \quad \nu_t(X_t(\varepsilon_t)-1)(1-\varepsilon_t) = z^{\nu_t} - \nu_t z. \end{aligned} \tag{37}$$

Implicitly differentiate (37) with respect to ε_t and, as z depends on ε_t, obtain

$$\frac{dX_t(\varepsilon_t)}{d\varepsilon_t}(1-\varepsilon_t) - (X_t(\varepsilon_t)-1) = (z^{\nu_t-1}-1)\frac{dz}{d\varepsilon_t} = (1+z+z^2+...+z^{\nu_t-2})(z-1)\frac{dz}{d\varepsilon_t}.$$

We now take the limits of the above-displayed equation and, as $\lim_{\varepsilon_t\to 0} X_t(\varepsilon_t) \to \frac{1}{\nu_t}$ from (37) and $\lim_{\varepsilon_t \to 0}(1+z+z^2+...+z^{\nu_t-2}) = \nu_t - 1$, obtain

$$\lim_{\varepsilon_t \to 0} \frac{dX_t(\varepsilon_t)}{d\varepsilon_t} = \frac{1}{\nu_t} - 1 - (\nu_t - 1)\lim_{\varepsilon_t \to 0}(1-z)\frac{dz}{d\varepsilon_t}.$$

The above-displayed equation, if $T-t=1$, by (35), gives

$$\lim_{\varepsilon_t \to 0} \frac{dX_t(\varepsilon_t)}{d\varepsilon_t} = \frac{1}{\nu_t} - 1 - (\nu_t - 1)\frac{1}{1-\nu_t} = \frac{1}{\nu_t},$$

and if $T-t \geq 2$, by (36), gives

$$\lim_{\varepsilon_t \to 0} \frac{dX_t(\varepsilon_t)}{d\varepsilon_t} = +\infty.$$

The proof is concluded observing that, for $\varepsilon_t = 0$, there is full-rent dissipation (as $\nu_t X_t(0) = 1$) and for $\varepsilon_t \to 0$ the aggregate bid strictly increases in ε_t. ■

Proof of Corollary 2. Call $\bar{\tau} \equiv \left\lceil 1 + \frac{\log\left(\frac{1}{2}\frac{\varepsilon_1}{1-\varepsilon_1}\right)}{\log\left(\frac{2}{3}\right)} \right\rceil$. By Proposition 7, for any ε_1, if $\bar{\tau} \leq T-1$, then we cannot have an all-in oligopoly from period $\bar{t} = \bar{\tau}$ onwards. By Proposition 5 and Proposition 1, the expected aggregate bid is smaller than 1 in any period $t \geq \bar{t}$. If instead $\bar{\tau} > T-1$, then we set $\bar{t} = T-1$ and note that, even if an all-in oligopoly occurs in period T, Proposition 3 implies that the last-period expected aggregate effort is $(1-\varepsilon_T)^{\nu_T - 1} < 1$. ■

Proof of Proposition 9. The proposition immediately follows noting that, in the equilibrium described in Proposition 3, all contestants without an all-in look bid 0 with certainty. ■

Type-asymmetric equilibria and casus irreducibilis: Section 6.

We focus on equilibria where, whenever contestants bid 1, they do so with identical probability. Hence, if any two rational contestants have an all-in look in a period, their all-in look (belief levels) is identical. In fact, recall from Section **6** that introducing in the model ex-ante asymmetries allows contestants to enter a certain period with different levels of (positive) beliefs, and this severely jeopardizes the tractability of our framework.

We consider asymmetric equilibria when $T=2$, $n=3$. When $\varepsilon=0$, all-in looks play no role and payoffs in $t=2$ are 0; in $t=1$, there is a continuum of asymmetric equilibria (without all-in efforts), as in BKD. Namely, two contestants mix in $[0,(1-p)^2]$ with CDF $x/(1-p)$ and in $[(1-p)^2, 1]$ with CDF $\sqrt{x}$. The third contestant plays 0 w.p. $1-p$ and with CDF $(\sqrt{x}-(1-p))/p$ on $[(1-p)^2, 1]$ w.p. p, where p is a free parameter. Similarly, when ε is sufficiently high, the fear of all-in automatons is high and thus all-in efforts are not played even in $t = 1$, so that payoffs in $t = 2$ are 0; in $t = 1$, there is a continuum of asymmetric equilibria (without all-in efforts), once again as in BKD. Namely, if $\varepsilon > (9 - \sqrt{33})/8$, two contestants mix in $[0, (1-\varepsilon)^2(1-p)^2]$ with CDF $x/((1-p)(1-\varepsilon)^2)$ and in $[(1-\varepsilon)^2(1-p)^2, (1-\varepsilon)^2]$ with CDF $(\sqrt{x} - (1-p)(1-\varepsilon))/((1-\varepsilon)p)$, while the third contestant plays 0 w.p. $1-p$ and uses CDF $(1-\varepsilon)\sqrt{x}$ on $[(1-\varepsilon)^2(1-p)^2, (1-\varepsilon)^2]$ w.p. p, where p is a free parameter.

Hence, we now focus on intermediate values of ε; namely $\varepsilon \in \left(0, \left(9 - \sqrt{33}\right)/8\right)$. Consider the following strategies in $t = 1$: all three contestants bid 1 w.p. q, contestants 1 and 2 play $F_1(x)$ on $x \in \left[0, \bar{d}\right]$ w.p. $1-q$, and contestant 3 plays $G_1(x)$ on $x \in \left[\underline{d}, \bar{d}\right]$ w.p. p and 0 w.p. $1-p-q$. Then, the equilibrium characterization in the main body implies that the payoffs in $t = 2$ are 0 unless $\nu_2 = 1$; in such case, the monopolist obtains $\varepsilon/\left(\varepsilon + (1-\varepsilon)q\right)$ by Bayes' rule and Proposition 2.

Contestant 1 (or, equivalently, contestant 2) by bidding $x \in \left[0, \underline{d}\right]$ obtains a cumulative payoff equal to $(1-\varepsilon)(1-q)F_1(x)(1-\varepsilon)(1-p-q) - x = 0$, whereas if she bids $x \in \left[\underline{d}, \bar{d}\right]$, she obtains $(1-\varepsilon)(1-q)F_1(x)\left((1-\varepsilon)(1-p-q) + (1-\varepsilon)p * G_1(x)\right) - x = 0$. Contestant 3 by bidding $x \in \left[\underline{d}, \bar{d}\right]$ obtains a cumulative payoff equal to $(1-\varepsilon)^2(1-q)^2 F_1(x)^2 - x = 0$. From those conditions, we obtain

$$F_1(x) = \begin{cases} \frac{x}{(1-\varepsilon)^2(1-q)(1-p-q)} & \text{if } x \in [0, \bar{d}] \\ \frac{\sqrt{x}}{(1-\varepsilon)(1-q)} & \text{if } x \in [\underline{d}, \bar{d}] \end{cases} \quad \text{and } G_1(x) = \frac{\sqrt{x} - (1-\varepsilon)(1-p-q)}{(1-\varepsilon)p} \text{ if } x \in [\underline{d}, \bar{d}].$$

Using $F_1(\bar{d}) = G_1(\bar{d}) = 1$ and $G_1(\underline{d}) = 0$ gives $\underline{d} = (1-\varepsilon)^2(1-p-q)^2$ and $\bar{d} = (1-\varepsilon)^2(1-q)^2$. Finally, any of the three contestants who bids 1 would obtain a cumulative payoff of[33]

$$(1-\varepsilon)^2(1-q)^2\left(1 + \frac{\varepsilon}{\varepsilon + (1-\varepsilon)q}\right) + 2(1-\varepsilon)(1-q)(\varepsilon + (1-\varepsilon)q) \cdot \frac{1}{2} + (\varepsilon + (1-\varepsilon)q)^2 \cdot \frac{1}{3} - 1 = 0,$$

which, by simple manipulations, can be written as $\tau(q, \varepsilon) \equiv Aq^3 + Bq^2 + Cq + D = 0$ where

$$A \equiv -(1-\varepsilon)^3, \; B \equiv 3(1-\varepsilon)^2(1-2\varepsilon), \; C \equiv 3(1-\varepsilon)\varepsilon(4-3\varepsilon), \; D \equiv -\varepsilon\left(4\varepsilon^2 - 9\varepsilon + 3\right).$$

The equation $\tau(q, \varepsilon) = 0$ has three roots in q. One of the three roots is always negative; in fact $\tau(0, \varepsilon) = D < 0$ by $\varepsilon \in \left(0, \frac{9-\sqrt{33}}{8}\right)$ and $\lim_{x \to -\infty} \tau(x, \varepsilon) = \infty > 0$. Another root is larger than 1; in fact $\tau(1, \varepsilon) = 2 > 0$ and $\lim_{x \to \infty} \tau(x, \varepsilon) = -\infty < 0$. The third root has $q \in (0, 1/2)$; in fact, $\tau(0, \varepsilon) = D < 0$ and $\tau(1/2, \varepsilon) = \frac{5+3\varepsilon+15\varepsilon^2-7\varepsilon^3}{8} > \frac{5}{8}$ by $\varepsilon \in (0, 1)$. Hence, in the only admissible solution, $q \in (0, 1/2)$ and we can choose $p \in (0, 1/2)$, so that $\underline{d} \in (0, 1)$. However, this unique admissible real root of $\tau(q, \varepsilon) = 0$ is a casus iriducibilis: it

[33] One can easily see that there are no profitable deviations for any contestant.

is real-valued but it cannot be expressed in radicals without using complex numbers. This can be seen as the discriminant of $\tau(q,\varepsilon)$ in q equals $27\varepsilon(1-\varepsilon)^6(12+3\varepsilon+8\varepsilon^2)>0$.

References

[1] Abreu, D., & Gul, F. (2000). Bargaining and reputation. *Econometrica*, 68(1), 85-117.

[2] Allison, J. R., Lemley, M. A., & Walker, J. (2010). Patent quality and settlement among repeat patent litigants. *Georgetown Law Journal*, 99, 677-712.

[3] Baye, M. R., Kovenock, D., & De Vries, C. G. (1996). The all-pay auction with complete information. *Economic Theory*, 8(2), 291-305.

[4] Beccuti, J. & Möller, M. (2022). Fighting for Lemons: The Encouragement Effect in Dynamic Contests with Private Information. Working Paper.

[5] Bliss, C., & Nalebuff, B. (1984). Dragon-slaying and ballroom dancing: The Private Supply of a Public Good. *Journal of Public Economics*, 25 (1), 1-12.

[6] Bulow, J. & Klemperer, P. D. (1999). The generalized war of attrition. *The American Economic Review*, 89, 175-189.

[7] Catepillan, J., Figueroa, N., & Lemus, J. (2022). Signaling in Dynamic Contests with Heterogenous Rivals. Available at SSRN.

[8] Cripps, Martin, Eddie Dekel, and Wolfgang Pesendorfer (2005). Reputation with Equal Discounting in Repeated Games with Strictly Conflicting Interests. *Journal of Economic Theory*, 121, 259-272.

[9] Dechenaux E., Kovenock, D., & Sheremeta, R. M. (2015). A survey of experimental research on contests, all-pay auctions and tournaments. *Experimental Economics*, 18, 609-669.

[10] Denter, P., Morgan, J., & Sisak, D. (2022). Showing off or laying low? The economics of psych-outs. *American Economic Journal: Microeconomics*, 14(1), 529-80.

[11] Donohue, J. J., & Levitt, S. D. (1998). Guns, violence, and the efficiency of illegal markets. *The American Economic Review P&P*, 88(2), 463-467.

[12] Ewerhart, C., & Lareida, J. (2024). Voluntary disclosure in asymmetric contests. Review of Economic Studies, 91, 2024, 3402-3422.

[13] Fu, Q. (2006). Endogenous timing of contest with asymmetric information. *Public Choice*, 129(1), 1-23.

[14] Fu, Q., Gürtler, O., & Münster, J. (2013). Communication and commitment in contests. *Journal of Economic Behavior & Organization*, 95, 1-19.

[15] Fu, Q. & Wu, Z. (2019). Contests: Theory and Topics. Oxford Research Encyclopedia of Economics. World Congress, J.-J. Laffont (ed.). Cambridge: Cambridge University Press.

[16] Fudenberg, D., Gilbert, R. , Stiglitz, J., & Tirole, J. (1983). Preemption, leapfrogging and competition in patent races. *European Economic Review*, 22 (1), 3-31.

[17] Fudenberg, D., & Tirole, J. (1986). A theory of exit in duopoly. *Econometrica*, 54 (4), 943-60.

[18] Fudenberg, D., & Tirole, J. (1991). Perfect Bayesian equilibrium and sequential equilibrium. *Journal of Economic Theory*, 53(2), 236-260.

[19] Gambetta, D. (2000). Mafia: the price of distrust. Trust: Making and breaking cooperative relations, 10, 158-175.

[20] Heijnen, P., & Schoonbeek, L. (2017). Signaling in a rent-seeking contest with one-sided asymmetric information. *Journal of Public Economic Theory*, 19(2), 548-564.

[21] Hörner, J., & Sahuguet, N. (2007). Costly signalling in auctions. *The Review of Economic Studies*, 74(1), 173-206.

[22] Hovenkamp, E. (2013). Predatory patent litigation: How patent assertion entities use reputation to monetize bad patents. Working Paper, Available at SSRN: 2308115.

[23] Katsenos, G. (2010). Long-term conflict: How to signal a winner?. University of Hannover, Working Paper.

[24] Konrad, K. A. (2004). Altruism and envy in contests: An evolutionarily stable symbiosis. *Social Choice and Welfare*, 22, 479-490.

[25] Konrad, K. (2009). Strategy and Dynamics in Contests. Oxford University Press

[26] Konrad, K. A., & Morath, F. (2018). To deter or to moderate? Alliance formation in contests with incomplete information. *Economic Inquiry*, 56(3), 1447-1463.

[27] Kovenock, D., Morath, F., & Münster, J. (2015). Information sharing in contests. *Journal of Economics & Management Strategy*, 24(3), 570-596.

[28] Krähmer, D. (2007). Equilibrium learning in simple contests. *Games and Economic Behavior*, 59(1), 105-131.

[29] Kreps, D. M., & Wilson, R. (1982). Reputation and imperfect information. *Journal of Economic Theory*, 27(2), 253-279.

[30] Kubitz, G. (2022). Two-Stage Contests with Private Information. Forthcoming at the *American Economic Journal: Microeconomics*.

[31] Kwiek, M. (2011). Reputation and cooperation in the repeated second-price auctions. *Journal of the European Economic Association*, 9(5), 982-1001.

[32] Levine, D. K. (2021). The reputation trap. *Econometrica*, 89(6), 2659-2678.

[33] Levitt, S. D., & Venkatesh, S. A. (2000). An economic analysis of a drug-selling gang's finances. *The Quarterly Journal of Economics*, (115)3, 755-789.

[34] Livingston, A. (2011). A reputation for violence. Colgate University. Available at: http://blogs.colgate.edu/lampert/wp-content/blogs.dir/39/files/2013/09/LivingstonA.pdf.

[35] Lugovskyy, V., Puzzello, D., & Tucker, S. (2010). An experimental investigation of overdissipation in the all pay auction. *European Economic Review*, 54, 974-997.

[36] MacCoun, R. J., & Reuter, P. (2001). Drug war heresies: learning from other times, vices and places. New York: Cambridge University Press.

[37] Mailath, G. J., & Samuelson, L. (2006). Repeated games and reputations: long-run relationships. Oxford University Press.

[38] Maynard Smith, J. (1974). The theory of games and the evolution of animal conflicts. *Journal of Theoretical Biology*, 47 (1), 209-21.

[39] Milgrom, P., & Roberts, J. (1982). Predation, reputation, and entry deterrence. *Journal of Economic Theory*, 27(2), 280-312.

[40] Münster, J. (2009). Repeated contests with asymmetric information. *Journal of Public Economic Theory*, 11(1), 89-118.

[41] Ordover, J. A., & Rubinstein, A. (1986). A sequential concession game with asymmetric information. *The Quarterly Journal of Economics*, 101 (4), 879-888.

[42] Shackelford, T. K., & Weekes-Shackelford, V. A. (Eds.). (2012). The Oxford handbook of evolutionary perspectives on violence, homicide, and war.

[43] Szech, N. (2011). Asymmetric all-pay auctions with two types. University of Bonn, Discussion paper, January.

[44] Tingley, D. H., & Walter, B. F. (2011). The effect of repeated play on reputation building: an experimental approach. *International Organization*, 65(2), 343-365.

[45] Wu, Z., & Zheng, J. (2017). Information sharing in private value lottery contest. *Economics Letters*, 157, 36-40.